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On the asymptotic behavior of the expectation of the maximum of i.i.d. random variables

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ABSTRACT

We study the asymptotic behavior of the expectation of the maximum of n i.i.d. random variables drawn from a fixed distribution F, with finite expectation. In this setting, Downey (1990) [4] showed that this expectation grows as $o(n)$. We provide an alternative simpler proof of Downey's result together with a tight lower bound.

1. Introduction

In this paper we study the asymptotic behavior of the expectation of the maximum order statistic $X_{(n)} = \max_{i=1,\dots,n} X_i$ of an independent sample $X_1, \dots, X_n$ drawn from a fixed distribution F, with finite expectation. The problem we address is to determine the worst case growth rate of $\mathbb{E}(X_{(n)})$ as n increases to infinity. This question is indeed quite natural and has been considered extensively in the applied probability and statistics communities in the past fifty years (see e.g. [1], [3], [4], [6]).

Note first that, since the maximum is upper bounded by the sum, we have that $\mathbb{E}(X_{(n)}) \le n\mathbb{E}(|X_1|)$, so $\mathbb{E}(X_{(n)}) = O(n)$. Similarly if $\mathbb{E}(X_1)^p < \infty$, it is easy to derive that $\mathbb{E}(X_{(n)}) = O(\sqrt[p]{n})$ using Jensen's inequality. Moreover, when the distribution F from where $X_1, \dots, X_n$ are drawn can depend on n, explicit bounds that depend on F where obtained by e.g. Arnold [1] and Downey [4], among others. However, when F is fixed and does not depend on n a much stronger and general bound can be obtained. Indeed, Downey [4] established that $\mathbb{E}(X_{(n)}) = o(n)$.[1]

Specifically, to establish that $\mathbb{E}(X_{(n)}) = o(n)$, Downey studies the sequence $X_{(n)}/\sqrt[p]{n}$, and uses a result by Freedman [5] to establish the convergence in probability of the sequence. Then he turns to prove that the sequence also converges in L^p. To this end, he shows that for $p = 1$ the sequence is uniformly integrable and thus, by Vitali convergence theorem (see e.g. [2, Theorem 4.5.4]), obtains L^1-convergence. For general $p \ge 1$, and under the assumption $\mathbb{E}(|X|^p) < \infty$, Downey uses Hölder inequality to reduce to the $p = 1$ case and concludes that $\mathbb{E}(|X_{(n)}|) = o(\sqrt[p]{n})$. Finally, Downey also argues that this bound is best possible in a certain sense.

In this note we present an elementary proof for the $o(n)$ bound for $\mathbb{E}(X_{(n)})$, which only uses the dominated convergence theorem and a basic calculus result. We also obtain Downey's result for general p as a corollary. Finally, we construct a lower bound that is stronger than Downey's, and conclude that $o(n)$ is indeed best possible.

2. Main result

In this section, we present our simple proof of Downey's result and use Jensen's inequality to state it in Downey's general form.

Theorem 1. *Let $X_1, \dots, X_n$ be independent random variables drawn from a common distribution F. Suppose $\mathbb{E}(|X_1|) < \infty$, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}(X_{(n)})}{n} = 0.$$

Proof. First note that it is enough to consider non-negative random variables, since $\mathbb{E}(X_{(n)}) \le \mathbb{E}\left(\max_{i=1,\dots,n} |X_i|\right)$. Now, as $X_{(n)} \sim F^n$ and it is also non-negative, its expectation can be written as

$$\mathbb{E}(X_{(n)}) = \int_0^\infty 1 - F^n(x)dx = \int_0^\infty (1 - F(x)) \sum_{k=0}^{n-1} F^k(x)dx.$$

The linearity of the integral implies that

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[1] More generally Downey establishes that if $\mathbb{E}(X_1)^p < \infty$ then $\mathbb{E}(X_{(n)}) = o(\sqrt[p]{n})$.

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$$\frac{\mathbb{E}(X_{(n)})}{n} = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^\infty F^k(x)(1-F(x))dx.$$

To conclude the proof recall that the arithmetic mean of a convergent sequence, converges to the same limit (see e.g. [7, Corollary 1.5]). Thus, to establish the theorem it is enough to argue that

$$\lim_{n\to\infty}\int_0^\infty F^k(x)(1-F(x))dx = 0.$$

This follows by the dominated convergence theorem since the sequence $(F^k(1-F))_{k\geq 0}$ converges pointwise to 0 and it is dominated by the integrable function $1-F$.[2] □

Note that by Vitali convergence theorem, the L^1-convergence of the sequence $(X_{(n)}/n)_{n\geq 1}$, is equivalent to its convergence in probability, and also to its uniform integrability. Therefore, the convergence in expectation we just showed also implies the convergence in probability and the uniform integrability shown by Downey. Furthermore, using Jensen's inequality for a convex function h, we get

$$h(\mathbb{E}(X_{(n)})) \leq \mathbb{E}(h(X_{(n)})) \leq \mathbb{E}\left(\max_{i=1,\dots,n} h(X_i)\right).$$

Thus we immediately obtain the following more general result:

Corollary 1.1. *For any convex function h, if $\mathbb{E}(h(X_1)) < \infty$, then $h(\mathbb{E}(X_{(n)})) = o(n)$. In particular, for all $p \geq 1$, if $\mathbb{E}(|X_1|^p) < \infty$, then $\mathbb{E}(X_{(n)}) = o(\sqrt[p]{n})$.*

3. Lower bound

Downey states that the bound $\mathbb{E}(X_{(n)}) = o(n)$ is best possible in the following sense. He proves that for all $\varepsilon > 0$ there exists a distribution F, such that $\mathbb{E}(X_{(n)}) = \Omega(n^{1-\varepsilon})$. However, this does not rule out the possibility of having a result stronger than that in Theorem 1, such as $\mathbb{E}(X_{(n)}) = O(n/\log(n))$. In this section, we argue that the bound from Theorem 1 is indeed best possible.

Theorem 2. *For any function g with sublinear growth, namely such that $g = o(n)$, there is a finite expectation distribution F such that if $X_1,\dots,X_n$ are independently drawn from F, then*

$$\liminf_{n\to\infty}\frac{\mathbb{E}(X_n)}{g(n)} > 0.$$

Proof. We establish the statement by constructing a distribution F such that for all sufficiently large n.

$$\mathbb{E}(X_{(n)}) \geq g(n).$$

For $k \geq 1$, define $a_k = g(k) - g(k-1)$. It is clear that $a_k \to 0$. We may assume wlog that $(a_k)_{k\geq 1}$ is a positive and non-increasing sequence. For otherwise we may take $\tilde{a}_k = \max_{m\geq k}(g(m)-g(m-1))$ and $\tilde{g}(n) = \sum_{k=1}^n \tilde{a}_k + g(0)$, which satisfies $\tilde{g}(n) \geq g(n)$ and $\tilde{g} = o(n)$.[3] Thus it is enough to show $\mathbb{E}(X_{(n)}) \geq \tilde{g}(n)$. Also, as we only need to show the inequality for sufficiently large n, we may assume that $a_k < 1$ for all $k \geq 1$ and that $g(0) \geq 0$.

Therefore we construct a distribution F of the form

$$F(x) = \sum_{k\geq 0}(1-a_k)\mathbf{1}_{I_k}(x),$$

for some disjoint intervals $I_k \subseteq \mathbb{R}$ with length $\delta_k \geq 0$. Letting $k(m) = \min\{k \geq 1 : a_k < 1/m\}$ for each $m \geq 1$, allows us to set

$$\delta_k = \begin{cases}\left(\frac{g(m)}{m} - \frac{g(m+1)}{m+1}\right)\frac{1}{a_{k(m)}}, & \text{if } k = k(m) \text{ for some } m \in \mathbb{N},\\ 0, & \text{otherwise.}\end{cases}$$

Note that since the sequence a_k is non-increasing we have that $\delta_k \geq 0$. To establish this we need to show that $(m+1)g(m) - mg(m+1) \geq 0$, which easily follows by induction because $g(0) \geq 0$.

With the previous choice of δ_k we immediately get that F has finite expectation. Indeed,

$$\int_0^\infty (1-F(x))dx = \sum_{k\geq 0} a_k\delta_k = \sum_{m\geq 1}\frac{g(m)}{m} - \frac{g(m+1)}{m+1} = g(1) < \infty.$$

On the other hand, if $Y_1,\dots,Y_n$ are independent random variables drawn from F, we have that

$$\mathbb{E}(Y_{(n)}) = \int_0^\infty (1-F^n(x))dx = \sum_{k\geq 0}(1-(1-a_k)^n)\delta_k.$$

To wrap up the proof we lower bound the latter expression. First recall that $(1-1/x)^x$ grows to e^{-1} as $x \to \infty$. Also, from the strict convexity of the exponential function, we have that if $x \in (0,1)$, then $\exp(-x) < 1-(1-1/e)x < 1-x/2$. Thus, since by definition of $k(n)$ we have that $0 < na_k < 1$, for all $k \geq k(n)$, we obtain

$$(1-a_k)^n = \left(\left(1-\frac{1}{1/a_k}\right)^{1/a_k}\right)^{na_k} \leq \exp(-na_k) < 1-na_k/2.$$

Putting all together we derive the lower bound

$$\mathbb{E}(Y_{(n)}) > \frac{n}{2}\sum_{k\geq k(n)} a_k\delta_k = \frac{n}{2}\sum_{m\geq n}\frac{g(m)}{m} - \frac{g(m+1)}{m+1} = \frac{1}{2}g(n),$$

which concludes the statement by taking $X_i = 2Y_i$. □

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[2] Note that the sequence actually decreases to 0, so monotone convergence can also be invoked.

[3] Indeed, $\lim_{n\to\infty}\frac{\tilde{g}(n)}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \tilde{a}_k = \lim_{n\to\infty}\tilde{a}_k = \limsup_{k\to\infty} g(k) - g(k-1) = 0.$