Posted Price Mechanisms and Optimal Threshold Strategies for Random Arrivals

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Abstract. The classic prophet inequality states that, when faced with a finite sequence of nonnegative independent random variables, a gambler who knows the distribution and is allowed to stop the sequence at any time, can obtain, in expectation, at least half as much reward as a *prophet* who knows the values of each random variable and can choose the largest one. In this work, we consider the situation in which the sequence comes in random order. We look at both a *nonadaptive* and an *adaptive* version of the problem. In the former case, the gambler sets a threshold for every random variable a priori, whereas, in the latter case, the thresholds are set when a random variable arrives. For the nonadaptive case, we obtain an algorithm achieving an expected reward within at least a 0.632 fraction of the expected maximum and prove that this constant is optimal. For the adaptive case with independent and identically distributed random variables, we obtain a tight 0.745-approximation, solving a problem posed by Hill and Kertz in 1982. We also apply these prophet inequalities to posted price mechanisms, and we prove the same tight bounds for both a nonadaptive and an adaptive posted price mechanism when buyers arrive in random order.

Keywords: optimal stopping • threshold rules • prophet inequality • posted price mechanisms • mechanism design • computational pricing and auctions

1. Introduction

One of the most basic economic problems is that of eliciting information to make optimal decisions. Consider, for instance, the fundamental problem of a seller who holds a single item and wants to sell it to buyers with private valuations for the item. The seller places no value on the item, whereas the buyers have independent, not necessarily identical, random valuations for the item. The main question is to design an incentive-compatible mechanism maximizing the revenue of the seller. This question was answered in a seminal paper by Myerson [35], and, as the solution is incentive compatible, it is in the buyers' best interest to declare their true valuations. Implementing this mechanism is remarkably simple in some situations. For instance, the optimal mechanism if the valuations are independent and identically distributed (i.i.d.) can be implemented as a second price auction with a common reserve price. However, this simple auction cannot always be implemented, for example, when not all prospective buyers are in the same place at the same time.

As implementing an auction is not always possible, a simple alternative to optimal auctions is provided by posted price mechanisms. In this setting, again we have a seller holding a single item to sell to a set of customers, who have independent random valuations for the item. Customers arrive sequentially, and the arrival sequence may be fixed, random, or chosen by the seller, depending on the specific context. In any case, upon arrival of a customer, the seller offers a price as a take-it-or-leave-it offer, and the customer either takes the item at that price or simply leaves it. The goal of the seller is to find the prices (and possibly the sequence of arrival) that maximize his expected revenue. These mechanisms are very appealing because of their simplicity and the fact that strategic behavior vanishes. So, quite naturally, they have been vastly studied, particularly in the marketing community (Bult and Wansbeek [8]). A common example of this practice is that of *direct mail campaigns*, in which the seller contacts his potential buyers directly and offers each one a certain

price for the item. The item is then sold to the first consumer who accepts the offer (Bult and Wansbeek [8], Cominetti et al. [13]).

Due to the relevance of these mechanisms in electronic commerce, several companies have started to apply *personalized pricing* to sell their products. Under this policy, companies set different prices for different consumers based on purchase history or other factors that may affect their willingness to pay. For example, the online data provider Lexis-Nexis sells to virtually every user at a different price (Shapiro and Varian [40]). In 2012, Orbitz online travel agency found that people who use Mac computers spent as much as 30% more on hotels, so it started to show them different, and sometimes costlier, travel options than those shown to Windows visitors (Mattioli [33]). Similarly, retailers and supermarket chains such as Safeway are using data culled from billions of purchases to offer deals tailored to specific shoppers (Kharif [27]). Recently, the Council of the European Union [36] has even agreed on regulation that will prevent such price discrimination based on nationality or place of residence.

Choudhary et al. [12] further investigated this issue, providing more examples and developing a theoretical framework to analyze equilibrium between firms that apply personalized pricing and those who do not.

In recent years, there has been a significant effort to understand the expected revenue of the outcome generated by different posted price mechanisms when compared with that of the optimal auction (Adamczyk et al. [2]; Blumrosen and Holenstein [7]; Chawla et al. [9],[10]; HajiAghayi et al. [21]; Yan [42]). Interestingly, first HajiAghayi et al. [21] and then Chawla et al. [10] noted a close connection between online mechanisms in general, and posted price mechanisms in particular, and *prophet inequalities*. These prophet inequalities measure the expected payoff of an all-knowing prophet relative to the best gambler in the theory of optimal stopping that was very active in the probability theory community three decades ago. Specifically, Chawla et al. [10] implicitly show that the problem of designing posted price mechanisms can be reduced to that of finding stopping rules of a related optimal stopping problem. Recently, Correa et al. [15] proved that the reverse direction also holds and therefore established an equivalence between designing posted price mechanisms and finding stopping rules for optimal stopping problems. This connection opened the way for new approaches and results and constitutes the starting point of this paper. We refer to the survey of Lucier [32] for further details.

1.1. Optimal Stopping Theory

Optimal stopping theory is concerned with choosing the right time to take a particular action, so as to maximize the expected reward. The famous prophet inequalities are a key example of a result in optimal stopping. There, a gambler faces a finite sequence of nonnegative independent random variables $X_1, \ldots, X_n$ with known distributions from which iteratively a prize is drawn. After seeing a prize, the gambler can either accept the prize and the game ends, or reject the prize and the next prize is presented to her. The classical result of Krengel and Sucheston [29, 30], also attributed to Garling, states that the gambler can obtain at least half of the expected reward that a prophet can make who knows the realizations of the prizes beforehand. That is, $\sup\{\mathbb{E}[X_t] : t \text{ stopping rule}\} \geq \mathbb{E}\{\sup_{1 \leq i \leq n} X_i\}/2$. Moreover, Krengel and Sucheston also showed that this bound is best possible. Samuel-Cahn [38] showed that the bound of 1/2 can be obtained by a good threshold rule, which stops as soon as a prize is above a fixed threshold. In later work, Samuel-Cahn [39] considers the case in which the random variables have a negative dependence. In this setting, she proves a slightly better bound and also shows that this bound is obtainable by the best threshold rule. Hill [22] studies the situation in which the order in which the random variables are presented can be chosen by the gambler. Kennedy [25], as well as Assaf et al. [4], considered the setting in which the gambler can select k different prizes. Whereas Kennedy looked at the situation in which the sum of the prizes is compared with that of the prophet, Assaf et al. studied the situation in which the maximum of the prizes is given as a reward. We refer the reader to the survey of Hill and Kertz [24] for more results on prophet inequalities. More recently, Esfandiari et al. [17] considered an interesting combination of the prophet inequality and the secretary problem, now known as the *prophet secretary problem*. This is basically a prophet inequality, but the the random variables are presented in a random order to the gambler. This setting provides the ground for the problems studied in this paper.

1.2. Problem Description

In the optimal stopping problem that we study, a gambler faces a sequence of n nonnegative independent random variables X_i with known distributions F_i for $i \in N = \{1, \ldots, n\}$. In this paper, we assume that the random variables arrive in a random order, where each order is equally likely. In every stage, a prize $\pi_i \sim F_i$ is drawn and the gambler needs to decide whether to accept and keep that prize, or to reject it and wait for the

next realization. The goal is to maximize his expected reward. We consider a nonadaptive and an adaptive scenario.

Nonadaptive. The gambler sets for each $i \in N$ a threshold $\tau_i \geq 0$, with the goal of maximizing his expected reward defined as

$$\mathbb{E}\left[\sum_{i \in N} \pi_i^\tau \mathbb{P}\left[i = \arg\min_{j \in N}\{\sigma(j) \mid \pi_j \geq \tau_j\}\right]\right],$$

where the probability is taken over the arrival permutation σ and the distributions F of the random variables. Furthermore, π_i^τ denotes the random variable $(\pi_i \mid \pi_i \geq \tau_i)$.

Adaptive. The gambler sets thresholds upon arrival of every random variable. So, the gambler sets functions $\tau_i : 2^N \to \mathbb{R}$ for each random variable X_i, such that, if S is the set of random variables that did not exceed their threshold before, then $\tau_i(S)$ is the threshold for random variable X_i if this is the next random variable to arrive. For an arrival permutation σ, we denote $\tau_i(\sigma) = \tau_i(\{\sigma^{-1}(1), \ldots, \sigma^{-1}(\sigma(i) - 1)\})$ and $\pi_i^\tau(\sigma) = (\pi_i \mid \pi_i \geq \tau_i(\sigma))$, and therefore we can write the gambler's expected revenue as

$$\mathbb{E}\left[\sum_{i \in N} \pi_i^\tau(\sigma) \mathbb{P}\left[i = \arg\min_{j \in N}\{\sigma(j) \mid \pi_j \geq \tau_j(\sigma)\}\right]\right],$$

where the expectation is taken over the arrival permutation σ, and the probability is taken over the distributions F of the random variables.

1.3. Related Work

For the nonadaptive version of the problem, we provide a stopping rule that guarantees an expected reward within a factor $1 - 1/e$ of the expected value of the maximum. Interestingly, this bound matches the bound of Chawla et al. [10] for the so-called *sequential posted price mechanisms* (SPM) in which the arrival order of the random variables is chosen by the gambler rather than at random. Furthermore, the bound also matches that of Esfandiari et al. [17]; however, their bound is obtained through a stopping rule that is adaptive.[1] Interestingly, subsequent to our work, there have been several improvements. First, Ehsani et al. [16] find yet another nonadaptive stopping rule with the same performance guarantee. Their stopping rule is essentially a fixed threshold stopping rule, though it requires randomized tiebreaking. This is indeed quite surprising, since their stopping rule is nonadaptive and anonymous. Furthermore, their result is best possible since, as we show in this paper, there does not exist a nonadaptive stopping rule achieving a guarantee on the expected reward within a factor strictly better than $1 - 1/e$ of the expected value of the maximum. Very recently, the factor $1 - 1/e$ for prophet secretary has been improved. Indeed, using a subtle analysis and case distinctions, Azar et al. [5] were able to design an algorithm achieving a ratio of $1 - 1/e + 1/400$. Furthermore, Correa et al. [14] obtained the currently best known guarantee by using randomized stopping rules based on the quantiles of the distribution of the maximum. In terms of upper bounds, Correa et al. [14] (in the full version of their paper) establish that no stopping rule can achieve a ratio better than $\sqrt{3} - 1 \approx 0.732$ for the prophet secretary problem.[2]

This latter upper bound, together with the fact that we show that the impossibility for nonadaptive stopping rules to beat $1 - 1/e$ holds even if the random variables are i.i.d., lead us to study adaptive stopping rules in this simpler setting. Of course, in the i.i.d. case, the classic prophet inequality and the prophet secretary problem are equivalent, and its study dates back to the work Hill and Kertz [23]. They provide a recursive characterization of the best possible a_n such that, if T_n is the set of stopping times of random variables $X_1, \ldots, X_n$, then

$$\mathbb{E}(\max\{X_1, \ldots, X_n\}) \leq a_n \sup\{\mathbb{E}(X_t) : t \in T_n\}. \tag{1}$$

They also prove that, for all $n > 1$, $1.1 < a_n < 1.6$, and they conjectured that the sequence a_n is monotone and that $a_n \leq 1 + 1/e \approx 1.368$. Shortly after, Samuel-Cahn [38] reported that Kertz [26] proved that $a_n \to \beta^* \approx 1.341 < \frac{1}{0.745}$ and conjectured that this limit constitutes the best possible upper bound (Saint-Mont [37] provides a simpler proof of this result). However, the best bound known until then was $a_n \leq e/(e - 1) \approx 1.582$ (Kertz [26, lemma 3.4]). Recently, and independent from our work, Abolhassani et al. [1] improved the bound

to 1.355. In this context, we study an adaptive stopping rule for i.i.d. distributions and prove that it leads to the optimal prophet inequality. Indeed, we show that its guarantee exactly matches the upper bound of $1/\beta^* \approx 0.745$ of Hill and Kertz [23], and Kertz [26]. It should be noted that this optimal prophet inequality was not only known in the limit when $n \to \infty$ (Kertz [26]), but also when the number of random variables is not fixed to n but is determined by a Poisson process (Allaart [3]). More recently, Kleinberg and Kleinberg [28] obtain the same bound in a continuous time setting, and Singla [41] provides a remarkably simple analysis for the Poisson arrival case.

Due to the reduction of Chawla et al. [10], our stopping rules carry over to posted price mechanisms. Thus, when the arrival order is random, we obtain a nonadaptive posted price mechanism obtaining an expected revenue within a factor $1 - 1/e$ of the expected revenue of the optimal auction. Also for i.i.d. valuations, we obtain an adaptive posted price mechanism that improves this bound to $1/1.341 > 0.745$.

It is worth mentioning that our work is indeed related to some of the literature in dynamic pricing. In this context, we typically have one or more units to sell to a number of strategic consumers with random valuation for a unit and that arrive according to a stochastic process, typically Poisson (Besbes and Lobel [6], Chen and Farias [11], Gallien [20]). In dynamic pricing, the goal of the seller is to come up with a price path, that is, a sequence of prices for the future, so as to maximize her revenue. Although the number of precise models is vast, a key difference between posted price mechanisms and dynamic pricing is that, in the former, the offers are take-it-or-leave-it, and thus strategic behavior vanishes, whereas, in the latter, buyers are long lived and forward looking. This means that a buyer may decide to wait and buy later so as to pay a lower price. In light of this, it is interesting to observe, for instance, that the fixed threshold algorithm of Ehsani et al. [16] for the prophet secretary problem does not translate into a fixed anonymous price in the setting of posted price mechanism, and thus it may be vulnerable to strategic behavior. The reason for this is that, to use a prophet inequality algorithm in the context of posted price mechanisms, one needs to go through Myerson's virtual values (see Section 5), and thus the *price* corresponding to a given threshold is given by the preimage of such a threshold through the virtual valuation (Chawla et al. [10], Correa et al. [15]). Thus, if the valuation distributions of different consumers are different, then the resulting prices will also be different.

1.4. Our Results

More formally, our nonadaptive stopping rule takes the form of the following result.

Theorem 1. *Given n independent nonnegative random variables $X_1, \ldots, X_n$ with $X_i \sim F_i$, there exist values $\tau_1, \ldots, \tau_n$ such that*

$$\mathbb{E}\left[\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}\right] \geq \left(1 - \frac{1}{e}\right) \mathbb{E}\left[\max_{i=1,\ldots,n}\{X_i\}\right],$$

where Y_i is a Bernoulli random variable that has value 1 if $X_i > \tau_i$. Here, when evaluating the expectation on the left-hand side, we define $0/0 = 0$.

As the X_i's are ordered uniformly at random, each variable X_i that is above τ_i, that is, when $Y_i = 1$, is equally likely to be the first X_i above τ_i. Hence, the expected reward of the gambler can be written as the quantity on the left.

The cornerstone of our analysis is a basic result about Bernoulli random variables that may be of independent interest. The result states that if we are given a set of nonhomogeneous independent Bernoulli random variables with associated prizes, then there is a subset of variables so that the expected average prize of the successes is at least a factor $1 - 1/e$ of the expectation of the maximum prize over all random variables.

Lemma 1 (Bernoulli Selection Lemma)**.** *Given a set $N = \{1, \ldots, n\}$ of independent Bernoulli random variables $Y_1, \ldots, Y_n$, where $Y_i = 1$ with probability q_i and 0 otherwise, and associated prizes $b_1, \ldots, b_n$, the following inequalities hold:*

$$\max_{S \subseteq N}\left\{\mathbb{E}\left[\frac{\sum_{i \in S} b_i Y_i}{\sum_{i \in S} Y_i}\right]\right\} \geq \left(1 - \frac{1}{e}\right) \max_{z_i \leq q_i}\left\{\sum_{i \in N} b_i z_i \;\middle|\; \sum_{i \in N} z_i \leq 1\right\} \geq \left(1 - \frac{1}{e}\right) \mathbb{E}\left[\max_{i \in N}\{b_i Y_i\}\right].$$

Here, when evaluating the leftmost term, we define $0/0 = 0$.

Note that, in Theorem 1, the rewards denoted by X_i are random variables, whereas, in the Bernoulli selection lemma, the prizes b_i are fixed. The lemma is related to online contention resolution schemes as described by Feldman et al. in [19, theorem 1.6].

To complement our results, we provide instances that show that the bounds in Lemma 1 and Theorem 1 are tight. In particular, we show that, even with i.i.d. random variables, the bound of Theorem 1 cannot be beaten. Therefore, to go beyond $1 - 1/e$, even when the random variables are i.i.d., a different setting needs to be considered. We examine the adaptive setting, and our adaptive stopping rule for the i.i.d. case takes the form of the following theorem.

Theorem 2. *Given nonnegative i.i.d. random variables $X_1, \ldots, X_n$, there exist thresholds $\tau_1, \ldots, \tau_n$, such that*

$$\mathbb{E}(\max\{X_1, \ldots, X_n\}) \leq \beta^* \mathbb{E}(X_t),$$

where $t := \min\{i \in \{1, \ldots, n\} : X_i \geq \tau_i\}$ and $\beta^* \approx 1.341 > \frac{1}{0.745}$ is the unique solution to

$$\int_0^1 \frac{1}{y(1 - \ln(y)) + (\beta - 1)} dy = 1. \tag{2}$$

Theorem 2 can be seen as a follow-up on a result by Hill and Kertz [23] on the prophet inequality for i.i.d. random variables. They study the performance of the best stopping time when compared with a prophet that can extract the expectation of the maximum. The main result of Hill and Kertz is a recursive characterization of a_n, the best possible factor when faced with n random variables. More precisely, they prove that if $X_1, \ldots, X_n$ are i.i.d. nonnegative random variables and T_n denotes the set of stopping rules for $X_1, \ldots, X_n$, then

$$\mathbb{E}(\max\{X_1, \ldots, X_n\}) \leq a_n \sup\{\mathbb{E}(X_t) : t \in T_n\}.$$

Furthermore, Hill and Kertz find instances in which it is not possible to beat the factor a_n. They also prove that $a_n \leq e/(e-1)$, conjecture that the sequence is monotone, and leave open the existence and computation of its limit. The monotonicity, together with the limit calculation, would readily give a universal bound (valid for all n) on the performance of the best stopping rule. Shortly after, Samuel-Cahn [38] reports that Kertz proves the existence of the limit a of the a_n sequence and conjectures that it equals 1.341 (obtained as the solution to $\int_0^1 (y - y \ln(y)) + a - 1)^{-1} dy = 1$). Finally, Kertz [26, lemma 6.2] proves the latter conjecture (for which Saint-Mont [37] derives a simpler proof). However, he is unable to prove that the sequence is monotone, and therefore the best upper bound on the whole a_n sequence still stood at $e/(e-1) \approx 1.582$ (Kertz [26, lemma 3.4]). Very recently, and independently of our work, Abolhassani et al. [1] improved this upper bound to $1/0.738 \approx 1.355$. Our Theorem 2 closes the gap and implies that, for all n, $a_n \leq a \approx 1.341$, and, by the tight examples of Hill and Kertz [23], this constant is best possible.

All results carry over to the setting of posted price mechanisms, and the corresponding corollaries and algorithms can be found in Section 5.

2. The Bernoulli Selection Lemma

In this section, we prove Lemma 1. We also provide a tight instance. To prove the lemma, we consider a continuous relaxation of the maximization problem and then guess a solution in which each random variable is included in S with some instance-dependent probability. Then, we look for the worst possible instance by applying the first-order optimality conditions of a nonlinear problem. These conditions reveal some structural insight on what a worst-case instance looks like. Using this, we obtain the desired bound.

The second inequality of Lemma 1 is trivial, as the expectation of the maximum is a sum over all values b_i weighted by the probability with which that value is the maximum. Since these probabilities sum to at most one, the inequality follows. Therefore, we only need to prove the first inequality, which is equivalent to

$$\max_{S \subseteq N} \left\{ \mathbb{E} \left[\frac{\sum_{i \in S} b_i Y_i}{\sum_{i \in S} Y_i} \right] \right\} \geq \frac{e-1}{e} \max_{z_i \leq q_i} \left\{ \sum_{i \in N} b_i z_i \; \middle| \; \sum_{i \in N} z_i \leq 1 \right\}. \tag{3}$$

The proof of this inequality has two main ingredients. First, we reformulate the left-hand side in an appropriate way, and lower-bound it by another function using Karush-Kuhn-Tucker (KKT) conditions. Then, we show that this function is bounded from below by $1 - 1/e$ times the maximization problem on the right-hand side. We note here that a simpler proof of Lemma 1 can be derived using a recent result of Ehsani et al. [16]. This alternative proof is presented in Appendix C.

A Simplified Proof. To provide the main intuition of the proof, we first show how to get a weaker result, that only gives us a factor of $1/\sqrt{e}$ instead of $(e-1)/e$, with more straightforward arguments.

Proof. We start the proof by rewriting the optimization problem:

$$\max_{S \subseteq N}\left\{\mathbb{E}\left[\frac{\sum_{i \in S} b_i Y_i}{\sum_{i \in S} Y_i}\right]\right\}. \tag{P}$$

Instead of choosing a subset of N, we set for each $i \in N$ a value $\chi_i \in [0,1]$, which represents the probability with which we actually choose i. Now, let $\pi_i = \chi_i q_i$ denote the probability of i being picked and having $Y_i = 1$. So, we can consider the following maximization problem, with decision variables π, as a relaxation of (P):

$$\max_{0 \le \pi_i \le q_i}\left\{\sum_{S \subseteq N}\left(\frac{b(S)}{|S|}\left(\prod_{i \in S}\pi_i\right)\left(\prod_{i \notin S}(1-\pi_i)\right)\right)\right\},$$

where $b(S) = \sum_{i \in S} b_i$. Note that, in this maximization problem, the objective function is linear in each π_i so that there is an extreme optimal solution. Thus, the previous problem is in fact equivalent to (P). Now, by changing the order of the summations, we obtain

$$\max_{0 \le \pi_i \le q_i}\left\{\sum_{i \in N}\left(b_i \pi_i \sum_{S \subseteq N \setminus \{i\}}\left(\frac{1}{1+|S|}\left(\prod_{j \in S}\pi_j\right)\left(\prod_{j \in N \setminus (S \cup \{i\})}(1-\pi_j)\right)\right)\right)\right\}. \tag{4}$$

Using this equivalent form, we can lower bound (P) by guessing a feasible solution. To this end, consider an optimal solution z^* to the maximization problem of the right-hand side in (3), that is, to

$$\max\left\{\sum_{i \in N} b_i z_i \;\middle|\; \sum_{i \in N} z_i \le 1, z_i \le q_i \text{ for all } i \in N\right\},$$

and set $\pi_i = z_i^*/(1 + z_i^*/2)$. Note that $\pi_i \le z_i^* \le q_i$, and thus it is feasible for (4). Moreover, as $1 - \pi_i = (1 - z_i^*/2)/(1 + z_i^*/2)$ and $\prod_{j \in S} z_j^* = 2^{|S|} \prod_{j \in S} z_j^*/2$, (P) can be lower-bounded by

$$\sum_{i \in N}\left(b_i z_i^*\left(\prod_{j \in N}\frac{1}{1+\frac{z_j^*}{2}}\right)\sum_{S \subseteq N \setminus \{i\}}\left(\frac{2^{|S|}}{1+|S|}\left(\prod_{j \in S}\frac{z_j^*}{2}\right)\left(\prod_{j \in N \setminus (S \cup \{i\})}\left(1 - \frac{z_j^*}{2}\right)\right)\right)\right).$$

It is easy to see that

$$\sum_{S \subseteq N \setminus \{i\}}\left(\frac{2^{|S|}}{1+|S|}\left(\prod_{j \in S}\frac{z_j^*}{2}\right)\left(\prod_{j \in N \setminus (S \cup \{i\})}\left(1 - \frac{z_j^*}{2}\right)\right)\right) \ge 1,$$

since the left-hand side corresponds to $\mathbb{E}[f(S)]$ over all $S \subseteq N \setminus \{i\}$ under probabilities $z_j^*/2$ for every element i and $f(S) = 2^{|S|}/(|S|+1) \ge 1$. Whereas for any values z_i such that $\sum_{i \in N} z_i \le 1$, we have

$$\prod_{j=1}^{n}\frac{1}{1+\frac{z_j}{2}} \ge e^{-\sum_{j=1}^{n}\frac{z_j}{2}} \ge \frac{1}{\sqrt{e}},$$

where the first inequality follows from $1 + x \le e^x$, concluding the proof. □

Obtaining a Bound of $1 - 1/e$. To obtain the factor $1 - 1/e$, we make a subtle modification in the choice of π_i. We will show that, for this choice of π_i, we can lower-bound (P) by $\sum_{i \in N} b_i z_i^* \varphi_{k(i)}(z_i^*)$, where $k(i)$, and thus $\varphi_{k(i)}(z_i^*)$ is a function that is only dependent on i. We then show that, for each i, $\varphi_{k(i)}(z_i^*) \ge 1 - 1/e$.

Thus, take $\pi_i = 2z_i^*/[2 + (e-2)z_i^*]$,[3] such that $1 - \pi_i = [2 - (4-e)z_i^*]/[2 + (e-2)z_i^*]$. Note that this is a feasible choice of π_i for all $i \in N$, since, for this choice, $\pi_i \leq z_i^* \leq q_i$.[4] We plug this back into (4) and obtain that (P) is lower-bounded by

$$\sum_{i \in N}\left(2b_iz_i^*\left(\prod_{j \in N}\frac{1}{2 + (e-2)z_j^*}\right)\sum_{S \subseteq N\setminus\{i\}}\left(\frac{2^{|S|}}{1 + |S|}\left(\prod_{j \in S}z_j^*\right)\left(\prod_{j \in N\setminus(S\cup\{i\})}\left(2 - (4-e)z_j^*\right)\right)\right)\right)$$

$$= \sum_{i \in N}b_iz_i^*\frac{2}{2 + (e-2)z_i^*}f_{N\setminus\{i\}}(z_{-i}^*), \tag{5}$$

where x_{-i} denotes the vector x with coordinate i eliminated and, for a given set M, $f_M(x)$ is defined as

$$f_M(x) = \left(\prod_{j \in M}\frac{1}{2 + (e-2)x_j}\right)\sum_{S \subseteq M}\left(\frac{2^{|S|}}{1 + |S|}\left(\prod_{j \in S}x_j\right)\left(\prod_{j \in M\setminus S}\left(2 - (4-e)x_j\right)\right)\right).$$

We find a lower bound on (5), by lower-bounding $f_{N\setminus\{i\}}(z_{-i}^*)$ by the minimum value for $f_{N\setminus\{i\}}(x)$ over all vectors x satisfying $\sum_{j \in N\setminus\{i\}}x_j \leq 1 - z_i^*$. According to Proposition A.1 in Appendix A, the minimum value is obtained by a solution x^* satisfying that all nonzero variables are equal and the sum of these variables is equal to $1 - z_i^*$.

Conditioning on the cardinality of the set S, and using the binomial theorem, a straightforward but tedious calculation shows that

$$f_{N\setminus\{i\}}(x^*) = \frac{2k + (e-2)(1 - z_i^*)}{2(k+1)(1 - z_i^*)}\left(1 - \left(1 - \frac{2(1 - z_i^*)}{2k + (e-2)(1 - z_i^*)}\right)^{k+1}\right).$$

As this quantity only depends on k and z_i^*, we may define

$$\varphi_k(z_i^*) = \frac{2}{2 + (e-2)z_i^*}f_{N\setminus\{i\}}(x^*),$$

to conclude that expression (5) [and in turn (P)] is lower-bounded by

$$\sum_{i \in N}b_iz_i^*\varphi_{k(i)}(z_i^*),$$

where the index $k(i) = |S|$ denotes the number of nonzero variables in x^* and is always at least 1, yet may vary depending on i.

Bounding φ by $1 - 1/e$. If $\varphi_{k(i)}(z_i^*) \geq 1 - 1/e$, then we can bound (P) from below by $\sum_{i \in N}b_iz_i^*\varphi_{k(i)}(z_i^*) \geq (1 - 1/e)$ $\sum_{i \in N}b_iz_i^*$ and the Bernoulli selection lemma is proved. Hereto, we show that, for any $n \geq 2$ and $y \in [0, 1]$, $\varphi_n(y) \geq 1 - \frac{1}{e}$. Recall that

$$\varphi_n(y) = \frac{2}{2 + (e-2)y}\frac{2n + (e-2)(1 - y)}{2(n+1)(1 - y)}\left(1 - \left(1 - \frac{2(1 - y)}{2n + (e-2)(1 - y)}\right)^{n+1}\right). \tag{6}$$

We start with a lemma that rephrases this claim. Since the proof basically consists of algebraic manipulations, we defer it to Appendix A.

Lemma 2. *Let $\varphi_n(y)$ be as defined in (6), and let*

$$h_n(x) := \frac{1}{n+1} - \frac{(1-x)^{n+1}}{n+1} - \frac{e-1}{2}x + \frac{(e-1)(e-2)n}{e(2 - (e-2)x)}x^2.$$

Then, $\varphi_n(y) \geq 1 - \frac{1}{e}$ for all $y \in [0, 1]$ and all $n \geq 2$ if and only if $h_n(x) \geq 0$ for all $n \geq 1$ and $x \in [0, \bar{x}]$, where $\bar{x} = 1/(n - 1 + e/2)$.

Hence, according to this lemma, we only need to prove that $h_n(x) \geq 0$ for all $n \geq 1$ and $x \in [0, \bar{x}]$. This is the subject of the following lemma, the proof of which can be found in Appendix A, for the same reason as for the proof of the previous lemma.

Lemma 3. *We have that $h_n(x) \geq 0$ for all $n \geq 1$ and $x \in [0, \bar{x}]$.*

From Lemmas 2 and 3, we conclude that indeed $\varphi_n(z_i^*) \geq 1 - 1/e$ for all $z_i^* \in [0, 1]$ and $n \geq 2$.

2.1. Tightness

We now provide a family of instances that shows that the $1 - 1/e$ bound in the Bernoulli selection lemma is actually best possible. First note that tightness with respect to the first inequality of the lemma is immediate by taking n Bernoulli random variables with parameter $1/n$ and all prizes equal to 1. To obtain tightness with respect to the second inequality, we need a more involved instance. Consider thus n^2 i.i.d. Bernoulli random variables with parameter $1/n$ and prizes $b_1 = n/(e-2)$ and $b_i = 1$ for $2 \le i \le n^2$. The expectation of the maximum prize is given by

$$\mathbb{E}\left[\max_{1 \le i \le n^2}\{b_i Y_i\}\right] = \frac{1}{e-2} + \left(1 - \frac{1}{n}\right)\left(1 - \left(1 - \frac{1}{n}\right)^{n^2-1}\right) \longrightarrow \frac{1}{e-2} + 1.$$

In this particular setting, where the Bernoulli random variables are i.i.d., the best strategy is to sort by prize and take some subset with those of higher prize. This means to choose the first random variable and a subset of size $k-1$ of the rest for some $1 \le k \le n^2$. This yields an expected prize that is equal to

$$\left(1 - \left(1 - \frac{1}{n}\right)^k\right)\frac{\frac{n}{e-2} + k - 1}{k} \le \left(1 - \left(1 - \frac{1}{n}\right)^k\right)\left(\frac{n}{k(e-2)} + 1\right).$$

Setting $x = k/n$, as $n \to \infty$, the aforementioned expression converges to

$$\max_{x \ge 0}(1 - e^{-x})\left(\frac{1}{x(e-2)} + 1\right).$$

Proposition A.3, in Appendix A shows that this expression is maximized at $x = 1$. This yields, for $n \to \infty$, the value $[1 - 1/e][1/(e-2) + 1] = (1 - 1/e)\mathbb{E}[\max_{1 \le i \le n^2}\{b_i Y_i\}]$.

3. Nonadaptive Threshold Rule

The Bernoulli selection lemma can be extended to more general random variables, namely, to the prophet inequality setting.

Theorem 1. *Given n independent nonnegative random variables $X_1, \ldots, X_n$ with $X_i \sim F_i$, there exist values $\tau_1, \ldots, \tau_n$ such that*

$$\mathbb{E}\left[\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}\right] \ge \left(1 - \frac{1}{e}\right)\mathbb{E}\left[\max_{i=1,\ldots,n}\{X_i\}\right],$$

where Y_i is a Bernoulli random variable that has value 1 if $X_i > \tau_i$. Here, when evaluating the expectation on the left-hand side, we define $0/0 = 0$.

Proof. Assume first that the F_i's are continuous for all i. Let $q_i = \mathbb{P}(X_i \ge X_j, \forall j = 1, \ldots, n)$ be the probability that X_i is the largest, and let α_i be a value for which $1 - F_i(\alpha_i) = q_i$. Consider $b_i = \mathbb{E}[X_i \mid X_i > \alpha_i]$ and the Bernoulli random variables $Z_1, \ldots, Z_n$, where Z_i has parameter q_i. We apply the Bernoulli selection lemma to this instance, and we thus let $S \subseteq \{1, \ldots, n\}$ be a set for which the lemma holds. Now define $\tau_i = \alpha_i$ for $i \in S$ and $\tau_i = \infty$ otherwise, and note that, for $i \notin S$, we have $Y_i = 0$ almost surely and that, for $i \in S$, we have $\mathbb{P}(X_i > \alpha_i) = \mathbb{P}(Y_i = 1) = q_i$. It follows that

$$\mathbb{E}\left[\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}\right] = \sum_{i \in S}\mathbb{E}\left[\frac{X_i Y_i}{\sum_{j \in S} Y_j}\right]$$

$$= \sum_{i \in S}\mathbb{E}[X_i \mid Y_i = 1]\mathbb{E}\left[\left(1 + \sum_{j \in S\setminus\{i\}} Y_j\right)^{-1}\bigg| Y_i = 1\right]\mathbb{P}(Y_i = 1)$$

$$= \sum_{i \in S}\mathbb{E}[X_i \mid X_i > \alpha_i]\mathbb{E}\left[\frac{Y_i}{\sum_{j \in S} Y_j}\right]$$

$$= \mathbb{E}\left[\frac{\sum_{i \in S}\mathbb{E}[X_i \mid X_i > \alpha_i]Z_i}{\sum_{i \in S} Z_i}\right]$$

$$\ge \frac{e-1}{e}\max_{z_i \le q_i}\left\{\sum_{i=1}^n \mathbb{E}[X_i \mid X_i > \alpha_i]z_i \bigg| \sum_{i=1}^n z_i \le 1\right\}$$

$$\ge \frac{e-1}{e}\sum_{i=1}^n \mathbb{E}[X_i \mid X_i > \alpha_i]q_i,$$

where the second-to-last inequality follows from the Bernoulli selection lemma, and the last holds since $\sum_{i=1}^{n} q_i = 1$. Now note that $\mathbb{E}[\max_{i=1,\dots,n}\{X_i\}] = \sum_{i=1}^{n} \mathbb{E}[X_i \mid X_i \geq X_j, \forall j = 1,\dots,n]q_i$. To finish the proof, it suffices to show that

$$\mathbb{E}[X_i \mid X_i > \alpha_i] \geq \mathbb{E}[X_i \mid X_i \geq X_j, \forall j = 1,\dots,n].$$

Indeed, if $x > \alpha_i$, then we have $\mathbb{P}(X_i > x \mid X_i > \alpha_i) = \int_x^{\infty}(1/q_i)dF_i(t)$, whereas, if $x \leq \alpha_i$, then this probability equals 1. On the other hand,

$$\mathbb{P}(X_i > x \mid X_i \geq X_j \, \forall j = 1,\dots,n) = \int_x^{\infty} \frac{\prod_{j\neq i} F_j(t)}{q_i} dF_i(t).$$

From this, it follows that $\mathbb{P}(X_i > x \mid X_i > \alpha_i) \geq \mathbb{P}(X_i > x \mid X_i \geq X_j, \forall j = 1,\dots,n)$ for all $x \geq 0$. Thus, $X_i \mid (X_i > \alpha_i)$ stochastically dominates $X_i \mid (X_i \geq X_j \, \forall j = 1,\dots,n)$, and the conclusion follows.

When some F_i are not continuous, it could be the case that there is no α_i such that $1 - F_i(\alpha_i) = q_i$ or that $\sum q_i > 1$. If the former happens, then the result still holds provided that α_i is chosen randomly. The latter case is solved by slightly perturbing the support of the random variables in a way that the probability that two or more are the maximum simultaneously is negligible. □

In the case of continuous distributions, the algorithm that achieves this result becomes remarkably simple. The algorithm, although randomized, can be derandomized using the method of conditional expectations (see, e.g., Motwani and Raghavan [34, section 5.6]).

Algorithm 1
Input: Random variables $X_i, i \in N$ with distributions F_i.
(1) Compute $q_i = \mathbb{P}(X_i = \max_{j\in N}\{X_j\})$.
(2) Set threshold $\tau_i = \begin{cases} F_i^{-1}(1 - q_i) & \text{with probability } 2/[2 + (e - 2)q_i], \\ \infty & \text{otherwise.} \end{cases}$
(3) Accept the first random variable having $X_i > \tau_i$.

To see that Algorithm 1 indeed yields an expected reward of at least $1 - 1/e$ of the reward of a prophet, we note that, in the proof of the Bernoulli selection lemma (Lemma 1), we set the variables $\pi_i = 2q_i/[2 + (e - 2)q_i]$. Therefore, the probability that we include a random variable in the set S is equal to $\chi_i = 2/[2 + (e - 2)q_i]$, which is exactly the probability that we set a finite threshold for variable X_i in step (2) of the algorithm. Furthermore, the thresholds α_i, defined in the proof of Theorem 1, are equal to $F^{-1}(1 - q_i)$ for continuous distributions.

Algorithm 1 may seem counterintuitive since, in step (2), the higher the probability is that a random variable is the maximum, the higher the probability is that the algorithm rejects it a priori (by setting a threshold of ∞), though the probability of rejecting any random variable is at most $1 - 2/e$. The following example gives some intuition as to why random variables need to be rejected. Consider just two random variables: X_1, which has deterministic value equal to 1, and X_2, which has a value of 100 with probability $1/10$ and 0 with probability $9/10$. In this situation, the expectation of the maximum equals $10 + 9/10$. Note that, in this instance, X_1 is much more likely to be the maximum; however, a nonadaptive algorithm that never discards it can get at most 50.5 when X_2 evaluates to 100 due to the random order. Therefore, it can get in total at most $(1/10)(50.5) + (9/10)(1) = 5.95$, which is not within the claimed ratio of the expectation of the maximum. Another somewhat surprising element of Algorithm 1 is that the probability of not accepting any random variable can be computed as $\prod_{i\in N}(1 - 2q_i/(2 + (e - 2)q_i)) \geq 2/e$. Again, the previous example provides intuition to the fact that, if we shoot for an algorithm that accepts too frequently, we risk settling for too low a prize. This intuition does not hold in the adaptive case.

3.1. Tight Instance with I.I.D. Distributions
We construct a family of instances with i.i.d. random variables, such that, for all $\varepsilon > 0$, there is an instance from this family for which no nonadaptive threshold rule can achieve an expected value within a factor $(1 + \varepsilon)(1 - 1/e)$ of the expected maximum. The idea is to mimic the instance that makes the Bernoulli selection lemma tight, but here we achieve this with i.i.d. distributions. Consider n^2 i.i.d. random variables following the law

$$X = \begin{cases} \frac{n}{e-2} & \text{w.p. } \frac{1}{n^3}, \\ 1 & \text{w.p. } \frac{1}{n}, \\ 0 & \text{w.p. } 1 - \frac{1}{n} - \frac{1}{n^3}. \end{cases}$$

Then, a prophet that can foresee all future prizes, obtains an expected reward of $(e-1)/(e-2)$ as $n \to \infty$. On the other hand, the best nonadaptive stopping rule sets a threshold of 1 for, say, random variables $X_1, \ldots, X_k$ and $n/(e-2)$ for the remaining random variables. The best value for k turns out to be roughly n (see Appendix B). The expected reward for this stopping rule approaches $[1/(e-2)+1][1-1/e]$ in the limit.

4. Adaptive Threshold Rule

In the previous section, we considered the setting in which the threshold value only depends on the random variable X_i, not on the order. Furthermore, we saw that, even when the distributions are i.i.d., no better factor than $1 - 1/e$ could be achieved under these assumptions. In this section, we consider the setting in which the threshold value may depend both on X_i and on the prizes that were rejected earlier. For i.i.d. random variables, we design an adaptive threshold strategy that achieves an expected revenue of at least a $1/\beta^* \approx 0.745$ fraction of the expected maximum value.

To achieve this result, we use a quite natural idea: we start with a high threshold, and, as fewer random variables are left, the thresholds will decrease. Besides this, the key ingredient of our algorithm is to use random thresholds drawn from a well-chosen distribution that mimics an expression that we obtain for the expected maximum value. In subsection "Quantile Stopping Rule," we describe the following algorithm that achieves the bound in Theorem 2 in the case of continuous distributions. Like Algorithm 1, this algorithm can also be derandomized using the method of conditional expectations (see, e.g., Motwani and Raghavan [34, section 5.6]).

Algorithm 2
Input: i.i.d. random variables X_i, $i \in N$ each with distribution F.
   (1) Partition the interval $[0,1]$ into intervals $A_i = [\varepsilon_{i-1}, \varepsilon_i]$, such that $\varepsilon_0 = 0$, $\varepsilon_n = 1$.
   (2) Sample q_i from A_i with an appropriately chosen distribution that only depends on n.[5]
   (3) When the i-th random variable comes, set threshold $\tau_i = F^{-1}(1 - q_i)$.

In the remainder of this section, we prove the bound of 0.745 for our adaptive threshold rule. Hereto, some notation is introduced, and this is summarized in Table 1.

For $X_1, \ldots, X_n$ nonnegative i.i.d. random variables, we take F as their probability distribution function and refer to X as a random variable with the same common distribution. Let $F^{-1}(q) = \inf\{x \geq 0 \mid F(x) \geq q\}$ be the generalized inverse of F (or quantile function), and let $\tau(q) = F^{-1}(1-q)$. Therefore, we have that $\mathbb{P}(X \geq \tau(q)) \geq q \geq \mathbb{P}(X > \tau(q)) = 1 - F(\tau(q))$, and this holds with equalities if F is continuous at $\tau(q)$. We start by deriving an expression for the expectation of the maximum of $X_1, \ldots, X_n$. Let $R(q) = \int_0^q F^{-1}(1-\theta)d\theta$, which, as we will see later, is equal to the expected reward from a random variable that is accepted with probability q. Now, we use Fubini's theorem and integration by parts on the product of $u(q) = (1-q)^{n-1}$ and $v'(q) = F^{-1}(1-q)$:

$$
\begin{aligned}
\mathbb{E}(\max\{X_1, \ldots, X_n\}) &= \int_0^\infty 1 - F^n(t)dt \\
&= \int_0^1 F^{-1}\left(\sqrt[n]{z}\right)dz \\
&= n \int_0^1 (1-q)^{n-1} F^{-1}(1-q)dq \\
&= n \int_0^1 (n-1)(1-q)^{n-2}\left(\int_0^q F^{-1}(1-\theta)d\theta\right)dq \\
&= n \int_0^1 (n-1)(1-q)^{n-2} R(q)dq.
\end{aligned}
\tag{7}
$$

Table 1. Notation used in Section 4.

$R(q) = \int_0^q F^{-1}(1-\theta)d\theta$	Expected reward of variable accepted with probability q
$A_i = [\varepsilon_{i-1}, \varepsilon_i]$	Interval for ith acceptance probability q_i
$\psi(q) = (n-1)(1-q)^{n-2}$	
$\gamma_i = \int_{q \in A_i} \psi(q)dq$	Normalization parameter
$f_i(q) = \psi(q)/\gamma_i$	Probability density function for choosing q_i
$\rho_1 = 1/\gamma_1$	
$\rho_{i+1} = (\rho_i/\gamma_{i+1})\int_{q \in A_i}\psi(q)(1-q)dq$	

Rather than directly constructing a threshold rule, our approach is to choose at each step a probability of acceptance, which naturally will be increasing as less random variables are left. One could compute the optimal threshold at each step. However, the analysis of such an optimal strategy becomes difficult. Here we use a less direct algorithm but obtain two big advantages: first, the thresholds that we end up with are explicit; and second, the posterior analysis becomes quite simple.

Specifically, when faced with a random variable X, we select a proper acceptance probability q. Now if F is continuous at $\tau(q)$, then we stop if $X \geq \tau(q)$ (so the acceptance probability is q). Otherwise (there is mass at $\tau(q)$), there may be no value that accomplishes the previous condition; so we stop if $X > \tau(q)$, and if $X = \tau(q)$, then we stop with probability $s = [q - \mathbb{P}(X > \tau(q))]/\mathbb{P}(X = \tau(q))$ (so again the acceptance probability is q). The goal behind this seemingly obscure rule is that, if at a given step we are faced with a random variable X and have decided on an acceptance probability q, then the expected reward in that step equals $R(q)$. Indeed, the reward can be calculated as

$$
\begin{aligned}
R(q) &= \mathbb{P}(X = \tau(q)) \cdot s \cdot \tau(q) + \mathbb{P}(X > \tau(q))\mathbb{E}[X|X > \tau(q)] \\
&= (q - \mathbb{P}(X > \tau(q)))\tau(q) + \mathbb{P}(X > \tau(q)) \int_0^\infty \mathbb{P}(X > t|X > \tau(q))dt \\
&= (q - \mathbb{P}(X > \tau(q)))\tau(q) + \mathbb{P}(X > \tau(q))\left(\tau(q) + \int_{\tau(q)}^\infty \mathbb{P}(X > t|X > \tau(q))dt\right) \\
&= q\tau(q) + \int_{\tau(q)}^\infty 1 - F(t)dt = \int_0^q F^{-1}(1 - \theta)d\theta,
\end{aligned}
$$

where the last equality follows from the definition of $\tau(q)$.

Quantile Stopping Rule. As described in Algorithm 2, our stopping rule is constructed as follows. We first partition the interval $A = [0, 1]$ into n intervals $A_i = [\varepsilon_{i-1}, \varepsilon_i]$, with $0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_{n-1} < \varepsilon_n = 1$. For random variable X_i, we draw an acceptance probability q_i at random from the interval A_i, according to the probability density function $f_i(q) = \psi(q)/\gamma_i$, where $\psi(q) = (n-1)(1-q)^{n-2}$ and γ_i is a normalization parameter equal to $\gamma_i = \int_{q \in A_i} \psi(q)dq$. As this q_i is the acceptance probability of variable X_i, the corresponding threshold at step i is $\tau_i = \tau(q_i)$. With all the q_i's at hand, we scan $X_1, \ldots, X_n$ and stop at step i with probability q_i using the previously described rule [i.e., if F is continuous at $\tau(q_i)$, then we stop if $X \geq \tau(q_i)$; otherwise, we stop for sure if $X > \tau(q_i)$, and if $X = \tau(q_i)$, then we stop with probability $s_i = [q_i - \mathbb{P}(X > \tau(q_i))]/\mathbb{P}(X = \tau(q_i))$].

The next two lemmas allow us to write the expected value of our algorithm as a constant (that only depends on n) times the expectation of the maximum X_i. In Lemma 4, we first write the value as the sum of the expected values that the algorithm would get from each of the X_i's, and Lemma 5 states that, by fine-tuning the ε_i's, we can obtain the desired result.

Lemma 4. *Let $\rho_1 = 1/\gamma_1$ and $\rho_{i+1} = (\rho_i/\gamma_{i+1}) \int_{\varepsilon_{i-1}}^{\varepsilon_i} \psi(q)(1-q)dq$ for $i = 1, \ldots, n-1$. Then the expected value of the random variable at which we stop, X_r, satisfies*

$$
\mathbb{E}(X_r) = \sum_{i=1}^n \rho_i \int_{\varepsilon_{i-1}}^{\varepsilon_i} (n-1)(1-q)^{n-2}R(q)dq.
$$

Proof. We have already shown that the expected value at step i equals $R(q_i)$. On the other hand, the probability that we get to step i is equal to $\prod_{j=1}^{i-1}(1 - q_j)$. Hence, by the linearity of expectation and independence of the q_i's, the expected value of X_r is

$$
\begin{aligned}
\mathbb{E}(X_r) &= \sum_{i=1}^n \mathbb{E}(R(q_i)) \prod_{j=1}^{i-1} \mathbb{E}(1 - q_j) \\
&= \sum_{i=1}^n \int_{\varepsilon_{i-1}}^{\varepsilon_i} (n-1)(1-q)^{n-2}R(q)dq \frac{\prod_{j=1}^{i-1} \int_{\varepsilon_{j-1}}^{\varepsilon_j} \psi(q)(1-q)dq}{\prod_{j=1}^i \gamma_i} \\
&= \sum_{i=1}^n \rho_i \int_{\varepsilon_{i-1}}^{\varepsilon_i} (n-1)(1-q)^{n-2}R(q)dq,
\end{aligned}
$$

where the last equality comes from the definition of ρ_i. $\square$

Our stopping rule still has freedom in the choice of $\varepsilon_1, \ldots, \varepsilon_{n-1}$. The next lemma tells us that an appropriate choice leads to an expression of the expected value of X_r in terms of the expectation of the maximum of $X_1, \ldots, X_n$.

Lemma 5. *If we choose $\varepsilon_1, \ldots, \varepsilon_{n-1}$ such that $\rho_1 = \rho_2 = \ldots = \rho_n$, then*

$$\mathbb{E}(\max\{X_1, \ldots, X_n\}) = n\gamma_1 \mathbb{E}(X_r).$$

Proof. If we choose $\varepsilon_1, \ldots, \varepsilon_{n-1}$ such that $\rho_i = \rho_1 = 1/\gamma_1$ for all i, then by (7) and Lemma 4 we can express

$$\mathbb{E}(\max\{X_1, \ldots, X_n\}) = n \int_0^1 (n-1)(1-q)^{n-2} R(q)\, dq$$

$$= n\gamma_1 \rho_1 \sum_{i=1}^n \int_{\varepsilon_{i-1}}^{\varepsilon_i} (n-1)(1-q)^{n-2} R(q)\, dq$$

$$= n\gamma_1 \sum_{i=1}^n \rho_i \int_{\varepsilon_{i-1}}^{\varepsilon_i} (n-1)(1-q)^{n-2} R(q)\, dq$$

$$= n\gamma_1 \mathbb{E}(X_r). \quad \square$$

Bounding γ_1. Since $\rho_{i+1} = (\rho_i/\gamma_{i+1}) \int_{\varepsilon_{i-1}}^{\varepsilon_i} \psi(q)(1-q)dq$ for all i, choosing $\varepsilon_1, \ldots, \varepsilon_{n-1}$ such that all ρ_i's are equal amounts to choosing them such that $\int_{\varepsilon_i}^{\varepsilon_{i+1}} \psi(q)dq = \int_{\varepsilon_{i-1}}^{\varepsilon_i} \psi(q)(1-q)dq$ for all i. By the definition of $\psi(q)$, this is equivalent to choosing them such that, for all i,

$$\frac{n-1}{n}\left((1-\varepsilon_{i-1})^n - (1-\varepsilon_i)^n\right) = (1-\varepsilon_i)^{n-1} - (1-\varepsilon_{i+1})^{n-1}.$$

Substituting $x_i = 1 - \varepsilon_i$, we obtain the following equivalent recursion on x_i:

$$\frac{x_{i-1}^n}{n} - \frac{x_i^n}{n} = \frac{x_i^{n-1}}{n-1} - \frac{x_{i+1}^{n-1}}{n-1}, \tag{8}$$

where $x_0 = 1$ and $x_n = 0$. With these boundary constraints, we can write this recursion as follows:

$$x_{i+1} = \left(\frac{n-1}{n} x_i^n - \frac{\alpha_n}{n}\right)^{1/(n-1)}. \tag{9}$$

When we substitute $\alpha_n = n - 1 - nx_1^{n-1}$, we have that (9) is the recursion from Hill and Kertz [23]. Indeed, for $i = 1$, Equation (8) gives

$$x_2 = \left(x_1^{n-1} + \frac{n-1}{n} x_1^n - \frac{n-1}{n}\right)^{1/(n-1)}. \tag{10}$$

Now, suppose that (9) holds for $i = 1, \ldots, j$. From (8), we have that

$$x_{j+1}^{n-1} = x_j^{n-1} + \frac{n-1}{n} x_j^n - \frac{n-1}{n} x_{j-1}^n$$

$$= \frac{n-1}{n} x_{j-1}^n + x_1^{n-1} - \frac{n-1}{n} + \frac{n-1}{n} x_j^n - \frac{n-1}{n} x_{j-1}^n$$

$$= \frac{n-1}{n} x_j^n + x_1^{n-1} - \frac{n-1}{n}.$$

Note that our quantity of interest γ_1 is equal to $\int_0^{\varepsilon_1} \psi(q)\, dq = 1 - x_1^{n-1}$. Therefore, if $n(1 - x_1^{n-1}) \le \beta^*$, then the expected value of the maximum is at most that of our stopping rule times β^*. Note that the value of x_1 (and all of the recursion) depends on n, but we have omitted this dependency for simplicity.

Observe that $n(1 - x_1^{n-1}) \le \beta$ is equivalent to $x_1 \ge (1 - \beta/n)^{1/(n-1)}$. Thus, if we find the minimum value of β such that $x_1 < (1 - \beta/n)^{1/(n-1)}$ implies $x_n < 0$, then we know that $x_1 \ge (1 - \beta/n)^{1/(n-1)}$ for that value of β. Hence, we proceed by showing an upper bound on the value of x_n.

Comparing this to Hill and Kertz [23], they prove that the smallest possible value a_n that satisfies their initial recurrence is equal to $1 + \alpha_n$, and therefore we can write a_n in terms of this new recursion as $a_n = n(1 - x_1{}^{n-1})$. By bounding γ_1, we thus also bound their quantity of interest and prove their conjecture.

Bounding the Recursion Through a Differential Equation. In the following, we show that each of the terms x_i in the recursion can be upper-bounded by a function $y(t) : [0, 1] \to \mathbb{R}$, defined through the differential equation:[6]

$$
\begin{aligned}
y'(t) &= y(t)\big(\ln(y(t)) - 1\big) - (\beta - 1), \\
y(0) &= 1.
\end{aligned} \tag{ODE}
$$

Furthermore, $y(1) := \lim_{t \uparrow 1} y(t)$ is the continuous extension of $y(t)$.

Later, we choose $\beta = \beta^* \approx 1.341$. For this β, we have $y \in [0, 1]$, and so we restrict our analysis of (ODE) to this interval. We assume that $\beta > 1.25$ and $y \in [0, 1]$. We validate these assumptions at the end of our analysis.

In the proof of Theorem 2, we use the following two lemmas. The proofs of these lemmas are deferred to Appendix D.

Lemma 6. *Differential equation* (ODE) *has a unique solution* $y(t)$, *which is a decreasing and strictly convex function on the interval* $[0, 1]$. *Furthermore,* $y'''(t) > 0$ *for* $y \in (0, 1)$.

The following lemma shows that the solution of (ODE) dominates the terms of the recurrence.

Lemma 7. *For* $x_1 < (1 - \beta/n)^{\frac{1}{n-1}}$, *we have* $x_i{}^{n-1} < y(i/n)$ *for* $i = 1, \ldots, n$, *where* $y(t)$ *is the unique solution of* (ODE).

Now we are ready to prove Theorem 2.

Theorem 2. *Given nonnegative i.i.d. random variables* $X_1, \ldots, X_n$, *there exist thresholds* $\tau_1, \ldots, \tau_n$, *such that*

$$
\mathbb{E}(\max\{X_1, \ldots, X_n\}) \le \beta^* \mathbb{E}(X_t),
$$

where $t := \min\{i \in \{1, \ldots, n\} : X_i \ge \tau_i\}$ *and* $\beta^* \approx 1.341 > 1/0.745$ *is the unique solution to*

$$
\int_0^1 \frac{1}{y\big(1 - \ln(y)\big) + (\beta - 1)} \, dy = 1. \tag{2}
$$

Proof. Consider the thresholds of the optimal threshold strategy, which can be easily computed by dynamic programming through the recurrence:

$$
\begin{cases}
\tau_n = 0, & V_n = \mathbb{E}(X) \\
\tau_i = V_{i+1}, & V_i = \mathbb{E}(X | X \ge \tau_i), \quad i = n-1, \ldots, 1.
\end{cases}
$$

Note that, under these thresholds, it is irrelevant whether to stop when $X \ge \tau_i$ or $X > \tau_i$, since they are constructed such that there is indifference between selecting a variable and keeping its value, or to continue for the expected value of the remaining variables. Therefore, any stopping rule that uses deterministic thresholds obtains an expected reward less than or equal to $\mathbb{E}(X_t)$.

We argue that the expected reward obtained by our randomized threshold rule $\mathbb{E}(X_r)$ is upper-bounded by the reward of a rule that uses only deterministic thresholds. Recall that our stopping rule randomizes at step i every time the corresponding threshold $\tau(q_i)$ has mass, choosing between accepting when $X > \tau(q_i)$ or $X \ge \tau(q_i)$. If we denote $\bar{q}_i = \mathbb{P}(X \ge \tau(q_i))$ and $\underline{q}_i = \mathbb{P}(X > \tau(q_i))$, then $F^{-1}(1 - q)$ is constant in $[\underline{q}_i, \bar{q}_i]$. Thus, $R(q) = \int_0^q F^{-1}(1 - \theta)d\theta$ is linear in that interval, implying that $E(X_r)$ is linear as a function of q_i when $\tau(q_i)$ has mass. This means that either the strategy that stops in step i whenever $X \ge \tau(q_i)$, or the strategy that does so when $X > \tau(q_i)$, attains a larger expected reward than $\mathbb{E}(X_r)$, and because both these strategies use only deterministic thresholds, they are in turn upper-bounded by $\mathbb{E}(X_t)$. Then, we know by Lemma 5 that

$$
\mathbb{E}(\max\{X_1, \ldots, X_n\}) \le n\gamma_1 \mathbb{E}(X_r) \le n\gamma_1 \mathbb{E}(X_t),
$$

where $\gamma_1 = 1 - x_1{}^{n-1}$.

It remains to show that $n(1 - x_1^{n-1}) \le \beta^*$ for $\beta^* \approx 1.341$. Lemma 7 yields $x_n < y(1)$, so we choose β such that $y(1) = 0$ to reach a contradiction with the fact that $x_n = 0$. Note that this indeed implies that $y \in [0, 1]$, as we assumed.

Hereto, note that $y(t)$ is invertible by Lemma 6. Hence, we can consider t as a function of y, for which we know that $t(1) = 0$, and we want to choose β such that $t(0) = 1$. In particular, we have that

$$t(1) = t(0) + \int_0^1 \frac{dt}{dy} dy = 1 + \int_0^1 \frac{1}{\frac{dy}{dt}} dy = 1 - \int_0^1 \frac{1}{y(1 - \ln(y)) + (\beta - 1)} dy.$$

So β^* is the value such that the last integral equals 1. This yields $\beta^* \approx 1.341$. □

Remark. A routine exercise shows that the sequence a_n defined by Hill and Kertz [23] exactly equals our $n\alpha_1$. Note here that our α_1 does depend on n, though we have omitted this dependency for simplicity of notation. Thus, our result implies that $a_n \leq \beta^*$ for all $n > 1$, and by the work of Kertz [26], we know that $a_n \to \beta^*$. Let T_n be the set of stopping rules for $X_1, \ldots, X_n$. Recalling that a_n is the smallest possible value for which

$$\mathbb{E}(\max\{X_1, \ldots, X_n\}) \leq a_n \sup\{\mathbb{E}(X_t) : t \in T_n\}, \tag{11}$$

we know that β^* is the smallest value satisfying (11) for all $n > 1$, and hence, it is tight.

5. Application to Posted Price Mechanisms
In this section, we discuss how our results can be applied to the setting of posted price mechanisms. In this setting, a seller has a single item to sell to a given set of customers $\mathcal{I}$. We assume that the seller has no value for keeping the item. Customers have independent random valuations for the item with customer $i \in \mathcal{I}$ valuing the item at v_i, drawn from distribution $F_i(\cdot)$. Customers arrive in (uniformly) random order, and, upon arrival of a customer, the seller offers a price as a take-it-or-leave-it offer, with the goal of maximizing his expected revenue. Similar to our optimal stopping problem, here we consider a nonadaptive and an adaptive scenario.

Nonadaptive. The seller sets prices $p_i \geq 0$ for all $i \in \mathcal{I}$, with the goal of maximizing his expected revenue, defined as

$$\sum_{i \in \mathcal{I}} p_i \mathbb{P}\left[i = \arg\min_{j \in \mathcal{I}} \{\sigma(j) \mid v_j \geq p_j\}\right],$$

where the probability is taken over the arrival permutation σ and the customers' valuations v.

Adaptive. The seller offers each customer a price as she arrives. So, the seller sets functions $p_i : 2^{\mathcal{I}} \to \mathbb{R}$ for each customer i, such that, if S is the set of customers who already arrived and declined the offer, then $p_i(S)$ is the price offered to customer i if she is next to arrive. For an arrival permutation σ, we denote $p_i(\sigma) = p_i(\{\sigma^{-1}(1), \ldots, \sigma^{-1}(\sigma(i) - 1)\})$, and therefore we can write the seller's expected revenue as

$$\mathbb{E}\left[\sum_{i \in \mathcal{I}} p_i(\sigma) \mathbb{P}\left[i = \arg\min_{j \in \mathcal{I}} \{\sigma(j) \mid v_j \geq p_j(\sigma)\}\right]\right],$$

where the expectation is taken over the arrival permutation σ, and the probability is taken over the customers' valuations v.

Our prophet inequalities translate into this posted price setting by a nice connection first made by Chawla et al. [10]. To briefly describe this connection, we first need a standard in the literature: we say that a valuation distribution F_i is *regular* if the *virtual value function* $c_i(v) = v - (1 - F_i(v))/f_i(v)$ is nondecreasing, where f_i is the density of F_i. A classic result of Myerson [35] states that the expected revenue of the optimal auction for selling a single item to the buyers in $\mathcal{I}$ equals the expectation of the maximum virtual value, $\mathbb{E}(\max_{i \in \mathcal{I}}\{c_i(v)\})$. On the other hand, if X is the positive part of the virtual valuation of a nonnegative random variable V with regular distribution F, then one can show that for $\tau \geq 0$ we have $\mathbb{E}(X \mid X > \tau) = F^{-1}(1 - q)$, where $q = \mathbb{P}(X > \tau)$. These two facts (with a slightly more involved version in case the distributions are not regular) amount to the conclusion that a comparison of posted price mechanisms to the optimal mechanism can be seen as a prophet inequality on the virtual values. Therefore, prophet inequalities can be applied in the context of posted price mechanisms and thus we obtain the following results.[7]

Corollary 1. *For any given set of potential customers $\mathcal{I}$, there exists a nonadaptive posted price mechanism that achieves an expected revenue of at least a $1 - 1/e$ fraction of that of Myerson's optimal auction on $\mathcal{I}$.*

Corollary 2. *For any given set of potential customers* $\mathcal{I}$ *whose values are independent and identically distributed, there exists an adaptive posted price mechanism that achieves an expected revenue of at least a* $1/\beta^* > 0.745$ *fraction of that of Myerson's optimal auction on* $\mathcal{I}$, *where* β^* *is the unique value satisfying* (2).

For the nonadaptive result, in the case of monotone virtual valuations, the mechanism can be directly derived from the threshold rule, and it becomes remarkably simple.

Algorithm 3

Input: Customers $i \in \mathcal{I}$ with valuation distributed according to F_i.
  (1) Compute $q_i =$ probability that optimal auction assigns to i.
  (2) Discard customer i with probability $1 - 2/[2 + (e-2)q_i]$.
  (3) Offer nondiscarded customers price $F_i^{-1}(1 - q_i)$.
  (4) Item is allocated to a random customer accepting the offer.

When valuations are nonregular, that is, the virtual values are not monotone, the price offered in step (3) is chosen randomly between two prices (see, e.g., Correa et al. [15]).

We remark that the tight instance constructed to prove the tightness of the Bernoulli selection lemma (Lemma 1) is also a valid instance for the nonadaptive setting of a posted price mechanism. The basic observation here is that, in that instance, the expectation of the maximum valuation and that of the optimal auction coincide, implying that the result from Corollary 1 is tight.

In the adaptive setting, our mechanism is as follows.

Algorithm 4

Input: Customers $i \in \mathcal{I}$ with valuation i.i.d. according to F.
  (1) Partition the interval $[0,1]$ into intervals $A_i = [\varepsilon_{i-1}, \varepsilon_i]$, such that $\varepsilon_0 = 0$, $\varepsilon_n = 1$.
  (2) Sample q_i from A_i with an appropriately chosen distribution.
  (3) When the i-th buyer comes, offer price $p_i = \max\{F^{-1}(1 - q_i), v^*\}$, where v^* is the reservation price of the optimal auction.

The assumption here is again that virtual valuations are monotone, and, as is the case for the nonadaptive algorithm, if the distribution is nonregular, then it suffices to set some prices as a lottery between two other prices.

Derivation of the Adaptive Posted Price Mechanism. Unlike the nonadaptive case, to derive the previous algorithm starting from the threshold rule is not as straightforward, due to the fact that we need to take a reservation price in the optimal auction into account. First we derive an exact expression for the expected revenue of the optimal auction for i.i.d. customers with cumulative distribution function $F(\cdot)$ and probability density function $f(\cdot)$. As in Myerson [35], we define the *virtual valuation* as $c(v) = v - [1 - F(v)]/f(v)$ and the *ironed virtual valuation* as $\bar{c}(v) = G'(F(v))$, where $G = \text{conv}(H)$ is the convexification of the negative revenue curve $H(q) = \int_0^q c(F^{-1}(\theta))d\theta$ as a function of the acceptance probability q. Let $\mathbb{E}(\text{MY}(n, F))$ be the expected revenue of the optimal auction over n customers with values drawn from distribution F.

Lemma 8. *For a given set of n i.i.d. potential customers with cumulative distribution function $F(\cdot)$, the expected revenue of Myerson's optimal auction is*

$$\mathbb{E}(\text{MY}(n, F)) = n(n-1) \int_0^1 (1-q)^{n-2} \bar{G}(1-q)dq, \tag{12}$$

where

$$\bar{G}(1-q) = \begin{cases} -G(1-q) & \text{if } 1-q > F(v^*), \\ v^*(1 - F(v^*)) & \text{otherwise.} \end{cases}$$

Proof. As mentioned earlier, the expected profit of the optimal auction equals its expected virtual surplus, that is, the sum over all customers of the expected value of the maximum of $\bar{c}$ above zero. Note that $\bar{c}$ is an increasing function, and let v^* be the value at which $\bar{c}(v^*) = 0$ or zero, if no such value exists. Then, the latter can be evaluated as

$$\mathbb{E}(\text{MY}(n, F)) = \int_{v^*}^{\infty} nF(v)^{n-1}\bar{c}(v)f(v)dv.$$

Performing the change of variables $q = 1 - F(v)$ and $\alpha^* = 1 - F(v^*)$, we obtain

$$
\begin{aligned}
\mathbb{E}(\text{MY}(n,F)) &= n \int_0^{\alpha^*} (1-q)^{n-1} \bar{c}(F^{-1}(1-q)) dq \\
&= n \int_0^{\alpha^*} (1-q)^{n-1} G'(1-q) dq \\
&= -nG(1-q)(1-q)^{n-1} \Big|_0^{\alpha^*} - \int_0^{\alpha^*} n(n-1)(1-q)^{n-2} G(1-q) dq \\
&= nG(1) - nG(F(v^*))F(v^*)^{n-1} - n(n-1) \int_0^{\alpha^*} (1-q)^{n-2} G(1-q) dq.
\end{aligned}
$$

Since $\bar{c}(v^*) = 0$, we know that G attains a minimum at $F(v^*)$ and, therefore, equals $H(F(v^*))$ at that point. Now, observe that

$$
H(q) = \int_0^q F^{-1}(\theta) - \frac{1-\theta}{f(F^{-1}(\theta))} d\theta = -(1-q)F^{-1}(q).
$$

Therefore, we can conclude that

$$
\begin{aligned}
\mathbb{E}(\text{MY}(n,F)) &= -nH(F(v^*))F(v^*)^{n-1} - n(n-1) \int_0^{\alpha^*} (1-q)^{n-2} G(1-q) dq \\
&= nv^*(1-F(v^*))F(v^*)^{n-1} - n(n-1) \int_0^{\alpha^*} (1-q)^{n-2} G(1-q) dq.
\end{aligned}
$$

Using the definition of $\bar{G}$, we can write the expected revenue of the optimal mechanism as (12), and the proof is complete. □

We note that expression (12), although fairly natural to derive from (7), appears to be new. In the adaptive setting, the price offered to every customer also depends on the set of customers that previously declined their offer. However, since the customers are i.i.d., it suffices to know only how many customers arrived before the current customer.

As in Section 4, we partition the interval $A = [0,1]$ into n intervals $A_i = [\varepsilon_{i-1}, \varepsilon_i]$ with $0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_n = 1$. We draw an acceptance probability q_i for the ith customer who arrives from interval A_i according to the probability density function $\psi(q_i) = (n-1)(1-q_i)^{n-2}/\gamma_i$, where γ_i is a normalization factor. We offer the customer a price of $\max\{F^{-1}(1-q_i), v^*\}$, where v^* is the reservation price of the optimal auction.

The expected revenue from selling the item to customer i is $\bar{G}(1-q_i)$. To see this, suppose that $q_i < 1 - F(v^*)$. Then, for monotone virtual valuations, the price offered to customer i is $F^{-1}(1-q_i)$, and thus the expected revenue is $q_i F^{-1}(1-q_i) = -G(1-q_i) = \bar{G}(1-q_i)$. On the other hand, if $q_i > 1 - F(v^*)$, then the price offered to customer i is v^*, which is accepted with probability $1 - F(v^*)$. Similar arguments hold when the virtual valuation is not monotone, where it might be the case that $q_i F^{-1}(1-q_i) = -H(1-q_i) < \bar{G}(1-q_i)$, and, by offering a price $F^{-1}(1-q_i)$, we might not get the best revenue. To circumvent this problem, we can randomize between two acceptance probabilities q_{i1} and q_{i2} such that $G(1-q_i) = \gamma H(1-q_{i1}) + (1-\gamma)H(1-q_{i2})$ and $q_i = \gamma q_{i1} + (1-\gamma)q_{i2}$.

Following the same reasoning as in Section 4, we can bound the expected revenue of this adaptive posted price mechanism by

$$
\sum_{i=1}^n \rho_i \int_{\varepsilon_{i-1}}^{\varepsilon_i} (n-1)(1-q)^{n-2} \bar{G}(1-q) dq,
$$

where $\rho_1 = 1/\gamma_1$ and $\rho_{i+1} = (\rho_i/\gamma_{i+1}) \int_{\varepsilon_{i-1}}^{\varepsilon_i} \psi(q)(1-q) dq$ for $i = 1, \ldots, n-1$. Again, if we choose $\varepsilon_1, \ldots, \varepsilon_{n-1}$ such that $\rho_1 = \rho_2 = \ldots = \rho_n$ and solve the recurrence on the ε_i values, then this quantity can be lower-bounded by

$$
\frac{1}{n\gamma_1} \mathbb{E}(\text{MY}(n,F)) \geq \frac{1}{\beta^*} \mathbb{E}(\text{MY}(n,F)) \approx 0.745 \mathbb{E}(\text{MY}(n,F)).
$$

Corollary 2 follows. As Correa et al. [15] show that the family of instances provided by Hill and Kertz [23] in the context of prophet inequalities for i.i.d. random variables can be transformed into a tight family of instances, the bound in Corollary 2 is tight.

6. Concluding Remarks

In this paper, we studied prophet inequalities in two contexts. First, we studied the prophet secretary problem in which the random variables are independent but distributed differently and are presented to the gambler in uniform random order. For this problem, we obtain a collection of thresholds, one for each random variable such that the associated stopping rule guarantees, in expectation, a value of at least $1 - 1/e$ for the expectation of the maximum value. In recent work, this bound has been beaten by Azar et al. [5] and further improved by Correa et al. [14]. Also, recent work of Eshani et al. [16] shows that the bound of $1 - 1/e$ can be obtained by a simpler, single-threshold stopping rule.

Then, we studied the i.i.d. prophet inequality, in which the random variables are independent and identically distributed. We obtain the best possible prophet inequality in this setting, solving an open problem of Hill and Kertz [23]. Interestingly, the optimal stopping rule can be obtained using a quantile strategy that, even before knowing the distribution, fixes the probabilities of stopping at each point in time.

An important problem that remains open is to find the optimal prophet inequality for the prophet secretary problem. Recently, it was discovered that the worst case for this problem is not the i.i.d. case (Correa et al. [14]), but it is perfectly plausible that the worst case consists of $n - 1$ i.i.d. random variables plus one deterministic one.

The main obstacle in applying our technique for the multiunit case is that we do not have the clean expression for the expectation of the maximum. This is key to our approach, as we basically use this seemingly new expression for the expectation of the maximum of n i.i.d. random variables to guide the decision rule at each step. This guidance is such that, overall, we recover a constant times the same area that is represented by the latter expectation. In the multiunit case, we lose this clean expression, and therefore our approach falls apart. Moreover, it is not even clear whether a quantile approach is best possible in this more general setting. Finally, it should be noted that this problem is essentially solved in the literature. Indeed, Ezra et al. [18] get a guarantee approaching to 1 exponentially fast as the number of items to select grows. This bound even works in the more general prophet secretary setting (not necessarily i.i.d.).

In the remainder of the paper, we present some technical results. First, in Appendix A, we present the many technicalities required to complete the picture of Section 2. In Appendix B, we analyze the tight instance for our approach for the prophet secretary problem with i.i.d. random variables. Later, in Appendix C, we present an alternative derivation of the Bernoulli selection lemma based on a recent result of Ehsani et al. [16], and, in Appendix D, we present the technical details concerning the ODE of Section 4.

Acknowledgments

Appendix A. Proofs and Propositions of Section 2

Lemma 2. *Let $\varphi_n(y)$ be as defined in (6), and let*

$$h_n(x) := \frac{1}{n+1} - \frac{(1-x)^{n+1}}{n+1} - \frac{e-1}{2}x + \frac{(e-1)(e-2)n}{e(2-(e-2)x)}x^2.$$

Then, $\varphi_n(y) \geq 1 - 1/e$ for all $y \in [0,1]$ and all $n \geq 2$ if and only if $h_n(x) \geq 0$ for all $n \geq 1$ and $x \in [0, \bar{x}]$, where $\bar{x} = 1/(n-1+e/2)$.

Proof. Consider the variable change

$$x = \frac{2(1-y)}{2(n-1) + (e-2)(1-y)},$$

so that

$$y = \frac{2 - (2(n-1)+e-2)x}{2 - (e-2)x}.$$

As y ranges from 0 to 1, x ranges from 0 to $1/(n-2+e/2)$. Note that

$$\frac{2}{2+(e-2)y} = \frac{2(2-(e-2)x)}{e(2-(e-2)x)-2(e-2)(n-1)x}.$$

Substituting this, we see that $\varphi_n(y) \geq (e-1)/e$ holds for all $y \in [0,1]$ and $n \geq 2$ if and only if

$$\frac{1}{n}(1-(1-x)^n) \geq \frac{e-1}{e} x \frac{e(2-(e-2)x)-2(e-2)(n-1)x}{2(2-(e-2)x)},$$

for all $x \in [0, 1/(n-2+e/2)]$ and $n \geq 2$. Moving the index of n by 1, the result follows. ☐

Lemma 3. *We have that $h_n(x) \geq 0$ for all $n \geq 1$ and $x \in [0, \bar{x}]$.*

Proof. We split the proof into the following parts, which together imply the result. All derivatives are with respect to x:
 (i) $h_n(0) = 0$ for all $n \geq 1$;
 (ii) $h_n(\bar{x}) \geq 0$ for all $n \geq 1$;
 (iii) $h_n'(0) > 0$ for all $n \geq 1$;
 (iv) $h_n'(\bar{x}) < 0$ for all $n \geq 1$;
 (v) $h_n'''(x) > 0$ for all $x \in [0, \bar{x}]$ and $n \geq 1$.
First, we show how the lemma follows from these parts (see also Figure A.1). Assume that (i)–(v) hold. We prove $h_n(x) \geq 0$ by contradiction. Assume that for some n there exists an $x_1 \in [0, \bar{x}]$ such that $h_n(x_1) < 0$. As $h_n(x)$ is differentiable and $h_n(\bar{x}) \geq h_n(0) = 0$, there exists an x_2 such that $h_n(x_2) < 0$ and $h_n'(x_2) = 0$. Since h_n increases from a negative value in x_2 to a nonnegative value in $\bar{x}$, there exists some $x_3 \in (x_2, \bar{x})$ such that $h_n'(x_3) > 0$. However, as $h_n'(\bar{x}) < 0$ and $h_n'(x_3) > 0$, there exists an $x_4 \in (x_3, \bar{x})$ with $h_n''(x_4) = 0$. By symmetry, the same analysis holds in the interval $(0, x_2)$ and, therefore, there also exists an $x_5 \in (0, x_2)$ with $h_n''(x_5) = 0$. However, this contradicts (v), as h_n'' is strictly increasing in x.

To prove the required statements, we compute the first three derivatives of f with respect to x:

$$h_n'(x) = (1-x)^n - \frac{e-1}{2} + \frac{n(e-1)(e-2)}{e}\left(\frac{4x-(e-2)x^2}{(2-(e-2)x)^2}\right),$$

$$h_n''(x) = n\left(-(1-x)^{n-1} + \frac{8(e-1)(e-2)}{e(2-(e-2)x)^3}\right),$$

$$h_n'''(x) = n\left((n-1)(1-x)^{n-2} + \frac{24(e-1)(e-2)^2}{(2-(e-2)x)^4}\right).$$

We finish the proof by proving the five statements:
 (i) We have that $h_n(0) = 0$ for all $n \geq 1$ by a direct calculation.
 (ii) From the proof of Lemma 2, it follows that $h_n(\bar{x}) \geq 0$ for all $n \geq 1$ is equivalent to $\varphi_n(z_i^*) \geq 1 - 1/e$ for $z_i^* = 0$ and all $n \geq 2$. By direct evaluation, we see that the latter is true for $n = 2, 3, 4$. Thus, it remains to show that, for all $n \geq 5$, we have

$$\frac{n-1+\frac{e-2}{2}}{n}\left(1-\left(1-\frac{1}{n-1+\frac{e-2}{2}}\right)^n\right) \geq 1 - \frac{1}{e},$$

Figure A.1. (Color online) Example function $h_n(x)$ with all implications of (i)–(v).
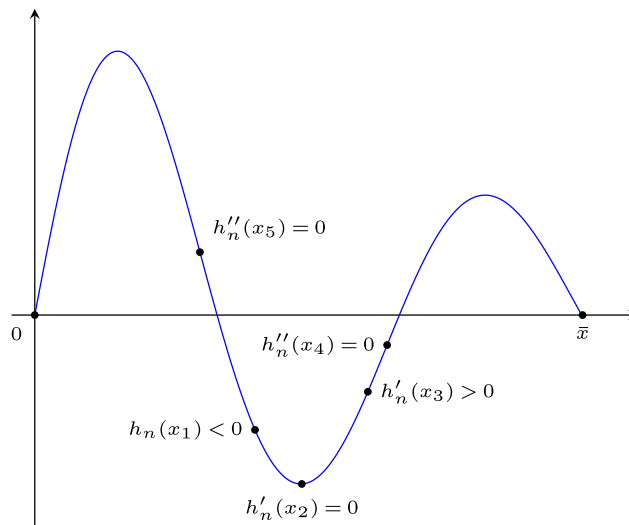
or, equivalently, that for all $n \geq 4$, we have

$$\frac{n-1+\frac{e}{2}}{n+1}\left(1-\left(1-\frac{1}{n-1+\frac{e}{2}}\right)^{n+1}\right) \geq 1-\frac{1}{e}.$$

We write this as

$$\left(1-\frac{1}{n-1+\frac{e}{2}}\right)^{n+1} \leq 1-\frac{(n+1)\left(1-\frac{1}{e}\right)}{n-1+\frac{e}{2}} = \frac{\frac{n+1}{e}+\frac{e}{2}-2}{n-1+\frac{e}{2}},$$

and multiplying both sides by $(1-1/(n-1+e/2))^{(e-5)/2}$ yields

$$\left(1-\frac{1}{n-1+\frac{e}{2}}\right)^{n+\frac{e}{2}-\frac{3}{2}} \leq \frac{\frac{n+1}{e}+\frac{e}{2}-2}{n-1+\frac{e}{2}}\left(1-\frac{1}{n-1+\frac{e}{2}}\right)^{\frac{e}{2}-\frac{5}{2}}$$

$$= \frac{\left(\frac{n+1}{e}+\frac{e}{2}-2\right)\left(n-1+\frac{e}{2}\right)^{\frac{3-e}{2}}}{\left(n-2+\frac{e}{2}\right)^{\frac{5-e}{2}}}.$$

According to Proposition A.2, the left-hand side is nondecreasing, and we see that it has limit $1/e$. For the right-hand side, note that the limit for n to infinity is also $1/e$, and the derivative with respect to n of the right-hand side is

$$4\frac{3-e}{e}(1-(3-e)n)\frac{(2n+e-2)^{\frac{1-e}{2}}}{(2n+e-4)^{\frac{7-e}{2}}},$$

which is negative for $n \geq 4$. The proof of (ii) is complete.

(iii) We have $h'_n(0) = 1 - (e-1)/2 > 0$.

(iv) For $n=1$ and $n=2$, direct evaluation of $h'_1(\bar{x})$ and $h'_2(\bar{x})$ gives negative values. For $n \geq 3$, proving that

$$h'_n(\bar{x}) = \left(1-\frac{1}{n+\frac{e}{2}-1}\right)^n + \frac{(e-1)\left((e-2)^2+2n(e-4)\right)}{4en} < 0,$$

which is equivalent to proving that

$$e\left(1-\frac{1}{n+\frac{e}{2}-1}\right)^n + \frac{(e-1)(e-2)^2}{4n} < \frac{(e-1)(4-e)}{2}.$$

According to Proposition A.2, $(1-1/(n+e/2-1)^n$ is nondecreasing, and by taking its limit and using $n \geq 3$ for the second term, we get

$$e\left(1-\frac{1}{n+\frac{e}{2}-1}\right)^n + \frac{(e-1)(e-2)^2}{4n} < 1 + \frac{(e-1)(e-2)^2}{12} < \frac{(e-1)(4-e)}{2} < 0.$$

(v) Since $0 \leq x \leq \bar{x} < 1$ for all n, $h'''_n(x)$ consists of sums, products, and quotients of only strictly positive terms. □

To complete the proof of the Bernoulli selection lemma, we need to prove Propositions A.1 and A.2.

Proposition A.1. *Consider the problem* $\min_{x\in\mathbb{R}_+^M}\{f_M(x) : \sum_{i\in M} x_i \leq a\}$, *where* $a \leq 1$ *and*

$$f_M(x) = \left(\prod_{j\in M}\frac{1}{2+(e-2)x_j}\right)\sum_{S\subseteq M}\left(\frac{2^{|S|}}{1+|S|}\left(\prod_{j\in S}x_j\right)\left(\prod_{j\in M\setminus S}(2-(4-e)x_j)\right)\right).$$

An optimal solution satisfies that all nonzero variables have to be equal and $\sum_{i\in M} x_i = a$.

Proof. Consider an optimal solution x^*, and assume its support is $M' \subseteq M$. Let y^* be the restriction of x^* to M'. Then, $f_M(x^*) = f_{M'}(y^*)$ and y^* minimizes $f_{M'}$. Consider the function $f(y_1, y_2)$ as the function $f_{M'}$ restricted to the first two variables, and the others are fixed to y_i^*. Clearly, y_1^*, y_2^* minimize $f(y_1, y_2)$ subject to the constraints that $y_1, y_2 > 0$, and $y_1 + y_2 \leq a - \sum_{i\in M'\setminus\{1,2\}} y_i^*$. Now $f(y_1, y_2)$ can be written as

$$f(y_1, y_2) = \frac{A + By_1 + By_2 + Cy_1y_2}{(2+(e-2)y_1)(2+(e-2)y_2)},$$

where

$$A = 4\left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|}}{1+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right),$$

$$B = \tfrac{e-4}{2}A + 2\left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|+1}}{2+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right),$$

$$C = \tfrac{e-4}{2}B + \left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|+2}}{3+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right),$$

with $N' = M' \setminus \{1,2\}$. Since the constraint $y_1 + y_2 \le a - \sum_{i\in N'} y_i^*$ is the only active constraint, and it is symmetric with respect to y_1 and y_2, the KKT conditions dictate that a minimum of $f(y_1, y_2)$ satisfies

$$\frac{\partial f(z_1^*, z_2^*)}{\partial z_1^*} = \frac{\partial f(z_1^*, z_2^*)}{\partial z_2^*}. \tag{A.1}$$

Taking the derivatives

$$\frac{\partial f(y_1, y_2)}{\partial y_1} = \frac{2B + 2y_2 C - (e-2)A - (e-2)y_2 B}{(2 + (e-2)y_1)^2 (2 + (e-2)y_2)},$$

$$\frac{\partial f(y_1, y_2)}{\partial y_2} = \frac{2B + 2y_1 C - (e-2)A - (e-2)y_1 B}{(2 + (e-2)y_1)(2 + (e-2)y_2)^2},$$

we see that (A.1) holds if and only if

$$\left(\left(4C - (e-2)^2 A\right) + \left(2(e-2)C - (e-2)^2 B\right)(y_2 + y_1) \right)(y_2 - y_1) = 0.$$

So either $y_1 = y_2$, or at least one is strictly positive and

$$y_1 + y_2 = \frac{(e-2)^2 A - 4C}{2(e-2)C - (e-2)^2 B}.$$

We evaluate the right-hand side of the latter, using the formulas for A, B, and C. Note first that $A \ge 0$, and observe that

$$B = \tfrac{e-4}{2}A + 2\left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|+1}}{2+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right)$$

$$\le \tfrac{e-4}{2}A + 4\left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|}}{2+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right)$$

$$\le \tfrac{e-4}{2}A + A = \tfrac{e-2}{2}A. \tag{A.2}$$

Now, note that $(2^{|S|+i+1})/(i+2+|S|) \ge (2^{|S|+i})/(i+1+|S|)$ for all $i, |S| \in \mathbb{R}_+$.

We can bound C by

$$C = \tfrac{e-4}{2}B + \left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|+2}}{3+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right)$$

$$\ge \tfrac{e-4}{2}B + \left(\prod_{j\in N'} \tfrac{1}{2+(e-2)y_j^*} \right) \sum_{S\subseteq N'} \left(\tfrac{2^{|S|+1}}{2+|S|} \left(\prod_{j\in S} y_j^* \right) \left(\prod_{j\in N'\setminus S} \left(2 - (4-e)y_j^*\right) \right) \right)$$

$$\ge \tfrac{e-4}{2}B + \tfrac{1}{2}B - \tfrac{(e-4)}{4}A$$

$$\ge \tfrac{e-3}{2}B - \tfrac{(e-4)}{4}A$$

$$\ge \tfrac{(e-3)(e-2)}{4}A - \tfrac{(e-4)}{4}A$$

$$= \tfrac{e^2 - 6e + 10}{4}A, \tag{A.3}$$

where the last inequality follows from (A.2). Now, using (A.2) and (A.3), we bound

$$
\begin{aligned}
y_1^* + y_2^* &= \frac{(e-2)^2 A - 4C}{2(e-2)C - (e-2)^2 B} \\
&\leq \frac{(e-2)^2 A - (e^2 - 6e + 10)A}{\frac{(e-2)(e^2-6e+10)}{2}A - \frac{(e-2)^3}{2}A} \\
&= \frac{(2e-6)A}{\frac{(e-2)((e^2-6e+10)-(e-2)^2)}{2}A} \\
&= \frac{2(2e-6)}{(e-2)(6-2e)},
\end{aligned}
$$

which is negative. This contradicts the constraint $y_1, y_2 > 0$. As the choice of the index $\{1,2\}$ is arbitrary, we conclude that all coordinates of y^* have to be equal.

To finish the proof, we still need to show that, in an optimal solution, the constraint $\sum_{i \in M} x_i^* = \sum_{i \in M'} y_i^* \leq a$ is tight. As all coordinates of y^* are equal, we know that $y_i^* = \bar{y}$, for some $\bar{y}$. Let k denote the number of nonzero variables in x^*, that is, $k = |M'|$. Then, abusing notation, we let

$$
\begin{aligned}
f_{M'}(\bar{y}) &= \left(\prod_{j \in M'} \frac{1}{2 + (e-2)\bar{y}} \right) \sum_{S \subseteq M'} \left(\frac{2^{|S|}}{1 + |S|} \left(\prod_{j \in S} \bar{y} \right) \left(\prod_{j \in M' \setminus S} (2 - (4-e)\bar{y}) \right) \right) \\
&= (2 + (e-2)\bar{y})^{-k} \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{2^\ell}{1+\ell} \bar{y}^\ell (2 - (4-e)\bar{y})^{k-\ell} \\
&= \frac{(2 + (e-2)\bar{y})^{k+1} - (2 - (4-e)\bar{y})^{k+1}}{(2 + (e-2)\bar{y})^k \cdot 2\bar{y}(k+1)}.
\end{aligned}
$$

We claim that $f_{M'}(\bar{y} + \varepsilon) \leq f_{M'}(\bar{y})$, for small $\varepsilon > 0$ and $\bar{y} < \frac{a}{k}$. Hereto, we take the derivative of $f_{M'}(\bar{y})$ with respect to $\bar{y}$ and show that this is nonpositive for $\bar{y} \geq 0$, from which the claim follows. After some tedious calculations, we have that

$$
\frac{\partial f_{M'}(\bar{y})}{\partial \bar{y}} = \frac{(2 + (e-2)\bar{y})^{-(k+1)}}{(k+1)\bar{y}^2} \left((2 - (4-e)\bar{y})^k (2 + (e-2)\bar{y} + 2k\bar{y}) - (2 + (e-2)\bar{y})^{k+1} \right).
$$

As $\bar{y} \geq 0$, it is easy to see that the sign of the derivative is equal to the sign of $(2 - (4-e)\bar{y})^k (2 + (e-2)\bar{y} + 2k\bar{y}) - (2 + (e-2)\bar{y})^k (2 + (e-2)\bar{y})$. Therefore, to show that the derivative is nonpositive for $\bar{y} \geq 0$, we need to show that

$$
(2 + (e-2)\bar{y})^{k+1} \geq (2 - (4-e)\bar{y})^k (2 + (e-2)\bar{y} + 2k\bar{y}). \tag{A.4}
$$

We prove this inequality by induction on k. For $k = 1$, we have

$$
\begin{aligned}
(2 + (e-2)\bar{y})^2 &= (2 - (4-e)\bar{y})(2 + (e-2)\bar{y} + 2\bar{y}) + 4\bar{y}^2 \\
&\geq (2 - (4-e)\bar{y})(2 + (e-2)\bar{y} + 2\bar{y}).
\end{aligned}
$$

Assume that (A.4) is true for given k. Then,

$$
\begin{aligned}
(2 + (e-2)\bar{y})^{k+2} &\geq (2 - (4-e)\bar{y})^k (2 + (e-2)\bar{y} + 2k\bar{y})(2 - (4-e)\bar{y} + 2\bar{y}) \\
&= (2 - (4-e)\bar{y})^{k+1} (2 + (e-2)\bar{y} + 2k\bar{y}) \\
&\quad + (2 - (4-e)\bar{y})^k (2 - (4-e)\bar{y} + 2(k+1)\bar{y})2\bar{y} \\
&\geq (2 - (4-e)\bar{y})^{k+1} (2 + (e-2)\bar{y} + 2k\bar{y}) + (2 - (4-e)\bar{y})^{k+1} 2\bar{y} \\
&= (2 - (4-e)\bar{y})^{k+1} (2 + (e-2)\bar{y} + 2(k+1)\bar{y}),
\end{aligned}
$$

where the first inequality is due to the induction hypothesis. Hence, (A.4) is true. For each $k \geq 1$ and $\bar{y} \geq 0$, the derivative is nonpositive, and $f_{M'}(\bar{y})$ is minimized for $\bar{y}$ as large as possible, that is, $\sum_{i \in M'} \bar{y} = a$. □

Proposition A.2. *Let* $c \in [0, \frac{1}{2}]$. *Then,* $f(x) = [1 - 1/(x+c)]^x$ *is a nondecreasing function of* x *in* $(1, \infty)$.

Proof. Define $g(x) = \ln(f(x)) = x \ln[1 - 1/(x+c)]$. We prove that $f(x)$ is nondecreasing by proving that $g'(x) \geq 0$. Note that

$$
g'(x) = \ln\left(\frac{x+c-1}{x+c} \right) + \frac{x}{(x+c-1)(x+c)},
$$

which is nonnegative if and only if

$$\frac{x}{(x + c - 1)(x + c)} \geq \ln\left(1 + \frac{1}{x + c - 1}\right).$$

We substitute $1/(x + c - 1) = z$. Thus, the right-hand side becomes $\ln(1 + z)$, and the left-hand side becomes

$$\frac{x}{(x + c - 1)(x + c)} = \frac{\frac{1}{z} - c + 1}{\frac{1}{z} + 1} z = \frac{1 + z - cz}{1 + z} z = \frac{z}{1 + z}(1 + (1 - c)z).$$

We expand $\ln(1 + z) = z - z^2/2 + z^3/3 - z^4/4 \pm \ldots$, so it is sufficient to prove

$$\frac{z}{1 + z}(1 + (1 - c)z) \geq z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \pm \ldots.$$

We multiply both sides by $(1 + z)/z$ to retrieve

$$1 + (1 - c)z \geq 1 + \frac{z}{2} - \frac{z^2}{6} + \frac{z^3}{12} - \frac{z^4}{20} \pm \ldots.$$

As $c \leq 1/2$, it suffices to prove $-z^2/6 + z^3/12 - z^4/20 \pm \ldots \leq 0$, that is,

$$\sum_{i=2}^{\infty} \frac{(-1)^i z^i}{i(i + 1)} \geq 0.$$

We rewrite this as

$$\sum_{i=2}^{\infty} \frac{1}{z} \frac{(-1)^i z^{i+1}}{i(i + 1)} = \sum_{i=2}^{\infty} \frac{1}{z} \int_0^z \frac{(-1)^i t^i}{i} dt = \frac{1}{z} \int_0^z \sum_{i=2}^{\infty} \frac{(-1)^i t^i}{i} dt,$$

so the proof is complete if we show that

$$\sum_{i=2}^{\infty} \frac{(-1)^i t^i}{i} \geq 0.$$

This is true, because

$$\sum_{i=2}^{\infty} \frac{(-1)^i t^i}{i} = -\sum_{i=1}^{\infty} \frac{(-1)^{i+1} t^i}{i} + t = -\ln(1 + t) + t \geq 0,$$

where the last inequality follows from $t \geq \ln(1 + t)$ for $t \geq 0$. $\quad \square$

Proposition A.3. *The function* $f(x) = [1 - e^{-x}][1/(x(e - 2)) + 1]$ *has a global maximum in* $x = 1$.

Proof. We compute the first two derivatives and find

$$f'(x) = \frac{-e^x + (e - 2)x^2 + x + 1}{(e - 2)x^2} e^{-x},$$

$$f''(x) = \frac{2e^x - ((e - 2)x^3 + x^2 + 2x + 2)}{(e - 2)x^3} e^{-x}.$$

We see that $f'(1) = 0$. To show that $x = 1$ is a global maximum, we prove that $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. To see this, first note that $f''(x)$ has the same sign as the function

$$g(x) = 2e^x - ((e - 2)x^3 + x^2 + 2x + 2).$$

Note further that $g(0) = 0$. Since this is an exponential function with a positive coefficient minus a polynomial with only positive coefficients, $g(x)$ first decreases until some point because of the polynomial, after which it is increasing because of the exponential term that starts to dominate the polynomial. So there exists some $x^* > 0$ such that $g(x) < 0$ for $x < x^*$, $g(x^*) = 0$, and $g(x) > 0$ for $x > x^*$. Since $g(1) = e - 3 < 0$, we know that $x^* > 1$. Therefore, $f''(x) < 0$ up to $x^* > 1$ and $f''(x) > 0$ afterward, and hence, $f'(x)$ is decreasing up to $x^* > 1$ and increasing afterward.

Since $f'(1) = 0$ and $f'(x)$ is decreasing for $x \leq 1$, we know that $f'(x) > 0$ for $x < 1$.

Furthermore, $f'(x) < 0$ for $1 < x \leq x^*$, since $f'(1) = 0$ and $f'(x)$ is decreasing for $1 < x < x^*$. Since $\lim_{x \to \infty} f'(x) = 0$ and $f'(x)$ is increasing from x^* onward, we know that $f'(x) < 0$ for $x > x^*$, and hence, $f'(x) < 0$ for all $x > 1$.

Therefore, $x = 1$ is a global maximum. $\quad \square$

Appendix B. Proof for the Optimal Nonadaptive Threshold Rule in Section 3.1

We now perform the analysis of the tight instance with i.i.d. distributions. We note first that, as n grows to infinity, the prophet's reward approaches $n/(e-2) \cdot (1/n) + 1 = (e-1)/(e-2)$. To upper-bound the expected reward of the best nonadaptive stopping rule, we condition on the number of random variables that have a high value and then estimate the right value of k. Thus, let V denote the random variable that has value equal to the reward that the stopping rule achieves. The expectation of V can be computed by conditioning on the number of random variables X_i that have a high value (i.e., value $n/(e-2)$). Denote the number of random variables X_i with high value by K; then,

$$\mathbb{E}[V] = \mathbb{P}[K = 0]\mathbb{E}[V|K = 0] + \mathbb{P}[K = 1]\mathbb{E}[V|K = 1] + \mathbb{P}[K \geq 2]\mathbb{E}[V|K \geq 2].$$

We bound each summand separately. The first term is straightforward:

$$\mathbb{P}[K = 0]\mathbb{E}[V|K = 0] = \left(1 - \frac{1}{n^3}\right)^{n^2}\left(1 - \left(\frac{1 - \frac{1}{n}}{1 - \frac{1}{n^3}}\right)^k\right)$$

$$\leq 1 - \left(1 - \frac{1}{n}\right)^k$$

$$\leq 1 - e^{-\frac{k}{n}}.$$

For the second term, we additionally condition on how many of the k selected random variables turn out to be 1:

$$\mathbb{P}[K = 1]\mathbb{E}[V|K = 1] \leq \left(n^2 \frac{1}{n^3}\left(1 - \frac{1}{n^3}\right)^{n^2}\right)\sum_{i=0}^{k}\frac{\frac{n}{e-2} + i}{i + 1}\binom{k}{i}\left(\frac{1}{n}\right)^i\left(1 - \frac{1}{n}\right)^{k-i}$$

$$\leq \frac{1}{n}\left(1 + \sum_{i=0}^{k}\frac{\frac{n}{e-2}}{i + 1}\binom{k}{i}\left(\frac{1}{n}\right)^i\left(1 - \frac{1}{n}\right)^{k-i}\right)$$

$$= \frac{1}{n}\left(1 + \frac{n^2}{(e-2)(k+1)}\sum_{i=1}^{k+1}\binom{k+1}{i}\left(\frac{1}{n}\right)^{i+1}\left(1 - \frac{1}{n}\right)^{k+1-i}\right)$$

$$= \frac{1}{n}\left(1 + \frac{n^2}{(e-2)(k+1)}\left(1 - \left(1 - \frac{1}{n}\right)^{k+1}\right)\right)$$

$$\leq \frac{1}{n}\left(1 + \frac{n^2}{(e-2)k}\left(1 - e^{-\frac{k}{n}}\right)\right)$$

$$\leq \frac{n}{(e-2)k}\left(1 - e^{-\frac{k}{n}}\right).$$

For the third term, the probability is so tiny that the obtained reward becomes irrelevant:

$$\mathbb{P}[K \geq 2]\mathbb{E}[V|K \geq 2] \leq \left(1 - \left(1 - \frac{1}{n^3}\right)^{n^2} - \frac{n^2}{n^3}\left(1 - \frac{1}{n^3}\right)^{n^2}\right)\frac{n}{e-2}$$

$$\leq \left(1 - \left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)\right)\frac{n}{(e-2)}$$

$$\leq \frac{1}{n(e-2)}.$$

Summing the three terms, and using Proposition A.3, we conclude that, for all $\varepsilon > 0$, there exists a large-enough n such that

$$\mathbb{E}[V] \leq \varepsilon + \max_x(1 - e^{-x})\left(1 + \frac{1}{x(e-2)}\right) \leq \left(1 - e^{-1}\right)\frac{e-1}{e-2} + \varepsilon.$$

Appendix C. Alternative Proof for the Bernoulli Selection Lemma

In this section, we present an alternative proof of the Bernoulli selection lemma, based on a recent result of Ehsani et al. [16].

Claim C.1 (Ehsani et al. [16, claim 23]). *Let $Z_1, \ldots, Z_n$ be random variables such that $\sum_i \mathbb{P}(Z_i > 0) \leq 1$. Then, there exists a threshold τ such that the expected value of a random variable chosen uniformly at random from the random set $R = \{i : Z_i \geq \tau\}$ is at least*

$$\left(1 - \frac{1}{e}\right) \sum_{i=1}^{n} \mathbb{E}[Z_i].$$

Using this claim, we can prove Lemma 1 as follows. Let z^* be the optimal (fractional) solution to $\max_{z_i \leq q_i} \{\sum_{i \in N} b_i z_i \mid \sum z_i \leq 1\}$. Define the random variables

$$Z_i = \begin{cases} b_i Y_i & \text{with probability } z_i^*/q_i, \\ 0 & \text{with probability } 1 - z_i^*/q_i, \end{cases}$$

which are well defined, as $z_i^* \leq q_i$. Also,

$$\sum_i \mathbb{P}(Z_i > 0) = \sum_i \mathbb{P}(Z_i = b_i) = \sum_i z_i^* \leq 1,$$

as z^* is feasible. We now proceed to use the previous claim. Let Z_τ be the value of the random variable chosen from the random set $R = \{i : Z_i \geq \tau\}$. Thus, there exists τ such that

$$\mathbb{E}[Z_\tau] \geq (1 - 1/e) \sum_i \mathbb{E}[Z_i] = (1 - 1/e) \sum_i b_i z_i^*.$$

In the original proof for the Bernoulli selection lemma, we argue that

$$\max_{S \subseteq N} \mathbb{E}\left[\frac{\sum_{i \in S} b_i Y_i}{\sum_{i \in S} Y_i}\right]$$

is in fact equivalent to the relaxation

$$\max_{0 \leq \pi_i \leq q_i} \left\{ \sum_{i \in N} \left(b_i \pi_i \sum_{S \subseteq N \setminus \{i\}} \left(\frac{1}{1 + |S|} \left(\prod_{j \in S} \pi_j \right) \left(\prod_{j \in N \setminus (S \cup \{i\})} (1 - \pi_j) \right) \right) \right) \right\}.$$

Now, let $T = \{i : b_i \geq \tau\}$ denote the set of random variables that can possibly cross τ. Set

$$\pi_i^* = \begin{cases} z_i^* & \text{if } i \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\sum_{i \in N} \left(b_i \pi_i^* \sum_{S \subseteq N \setminus \{i\}} \left(\frac{1}{1 + |S|} \left(\prod_{j \in S} \pi_j^* \right) \left(\prod_{j \in N \setminus (S \cup \{i\})} (1 - \pi_j^*) \right) \right) \right)$$

$$= \sum_{i \in T} \left(b_i z_i^* \sum_{S \subseteq T \setminus \{i\}} \left(\frac{1}{1 + |S|} \left(\prod_{j \in S} z_j^* \right) \left(\prod_{j \in T \setminus (S \cup \{i\})} (1 - z_j^*) \right) \right) \right)$$

$$= \sum_{i \in T} b_i z_i^* \sum_{S \subseteq T \setminus \{i\}} \frac{1}{1 + |S|} \mathbb{P}(S = R \setminus \{i\})$$

$$= \sum_{i \in T} \mathbb{E}[Z_i \mid Z_i = b_i] \mathbb{P}(Z_i = b_i) \cdot \mathbb{P}(i \text{ is chosen at random from } R \cup \{i\})$$

$$= \sum_{i \in T} \mathbb{E}[Z_i \mid i \in R] \mathbb{P}(i \in R, i \text{ is chosen at random from } R)$$

$$= \mathbb{E}[Z_\tau],$$

where the second and second-to-last equalities follow due to the fact that, for any $i \in T$, $Z_i = b_i$ is equivalent to $Z_i \geq \tau$ (or $i \in R$).

Appendix D. Proofs and Proposition for Section 4

Lemma 6. *Differential equation (ODE) has a unique solution $y(t)$, which is a decreasing and strictly convex function on the interval $[0, 1]$. Furthermore, $y'''(t) > 0$ for $y \in (0, 1)$.*

Proof. Note that $y'(0) = -\beta < 0$ because $y(0) = 1$. For $y \in (0, 1]$, we know that $\ln(y) \leq 0$. Also, as $\beta > 1$, we conclude that $y'(t) < 0$. Furthermore, $y(t)$ is convex, as for $y \in [0, 1)$,

$$y'' = y'(\ln(y) - 1) + y\frac{y'}{y} = y'\ln(y) > 0,$$

and $y'' = 0$ for $y = 1$. Finally,

$$y''' = y''\ln(y) + y'\frac{y'}{y} = y'\ln^2(y) + \frac{(y')^2}{y} = y'\left(\ln^2(y) + \ln(y) - 1 - \frac{\beta - 1}{y}\right).$$

We show that $\ln^2(y) + \ln(y) - 1 - (\beta - 1)/y < 0$ for $y \in (0, 1)$ or, equivalently, that $g(y) = y\ln^2(y) + y\ln(y) - y - \beta + 1 < 0$ for $y \in (0, 1)$. To determine the maximum value of $g(y)$, observe that

$$\frac{dg(y)}{dy} = \ln^2(y) + 2y\ln(y)\frac{1}{y} + \ln(y) + y\frac{1}{y} - 1$$
$$= \ln^2(y) + 3\ln(y) = \ln(y)(\ln(y) + 3).$$

Note that $dg(y)/dy \geq 0$ on $y \in (0, e^{-3})$ and $g'(y) < 0$ on $y \in (e^{-3}, 1)$. Hence, since $g(y)$ is continuous, its maximum is attained at $y = e^{-3}$, and $g(e^{-3}) = 5e^{-3} - \beta + 1 < 0$, as $\beta > 1.25$.

Moreover, note that if $y \in (0, 1)$, then $|y''|$ is bounded, and hence y' is Lipschitz continuous. Therefore, by the Picard-Lindelöf theorem (Lindelöf [31]), $y(t)$ is unique on $(0, 1)$. As $y(0)$ is given, and we defined $y(1)$ as the continuous extension of $y(t)$, the solution $y(t)$ is unique on $[0, 1]$. $\quad\square$

Lemma 7. *For $x_1 < (1 - \beta/n)^{1/(n-1)}$, we have $x_i^{n-1} < y(i/n)$ for $i = 1, \ldots, n$, where $y(t)$ is the unique solution of* (ODE).

Proof. First note that $x_0 = y(0) = 1$, by definition. Moreover, we already saw that $y'(0) = -\beta$. As $y(t)$ is strictly convex and since $x_1 < (1 - \beta/n)^{1/(n-1)}$, we have that $y(1/n) > y(0) - \frac{1}{n}\beta > x_1^{n-1}$, proving the statement for $i = 1$. We proceed by induction, assuming that $x_i^{n-1} < y(\frac{i}{n})$. Taylor's theorem states that there exists $\zeta \in [i/n, (i+1)/n]$ such that

$$y\left(\frac{i+1}{n}\right) = y\left(\frac{i}{n}\right) + \frac{1}{n}y'\left(\frac{i}{n}\right) + \frac{1}{2n^2}y''\left(\frac{i}{n}\right) + \frac{1}{6n^6}y'''(\zeta).$$

The previous expression follows from the mean value form of the remainder of the Taylor expansion of $y((i+1)/n)$ around i/n. Now,

$$y\left(\frac{i+1}{n}\right) > y\left(\frac{i}{n}\right) + \frac{1}{n}y'\left(\frac{i}{n}\right) + \frac{1}{2n^2}y''\left(\frac{i}{n}\right) = y\left(\frac{i}{n}\right) + \frac{y'\left(\frac{i}{n}\right)}{n}\left(1 + \frac{\ln(y(\frac{i}{n}))}{2n}\right)$$
$$= y\left(\frac{i}{n}\right) + \frac{y(\frac{i}{n})\left(\ln(y(\frac{i}{n})) - 1\right) - (\beta - 1)}{n}\left(1 + \frac{\ln(y(\frac{i}{n}))}{2n}\right)$$
$$\geq \frac{n-1}{n}y\left(\frac{i}{n}\right)^{\frac{n}{n-1}} - \frac{\beta - 1}{n} > \frac{n-1}{n}x_i^n - \frac{\beta - 1}{n} > x_{i+1}^{n-1}.$$

Here, the first inequality follows from $y''' > 0$, the second from Proposition D.1, the third from the induction hypothesis, and the last from Equation (9) and the assumption that $x_1^{n-1} < 1 - \beta/n$. $\quad\square$

To finalize the proof of Lemma 7, we show the following proposition.

Proposition D.1. *For $x \in (0, 1]$ and $n \geq 2$, we have*

$$x + \frac{x(\ln(x) - 1)}{n} + \frac{\ln(x)\left(x(\ln(x) - 1) - (\beta - 1)\right)}{2n^2} \geq \frac{n-1}{n}x^{\frac{n}{n-1}}.$$

Proof. Fix a value for n. Since $-(\beta - 1)\ln(x)$ is nonnegative, and since $x > 0$, it suffices to prove that

$$f(x) := 1 + \frac{\ln(x) - 1}{n} + \frac{\ln(x)(\ln(x) - 1)}{2n^2} - \frac{n-1}{n}x^{\frac{1}{n-1}} \geq 0.$$

As $f(1) = 0$ for all n, showing that f is nonincreasing completes the proof. We have that $f'(x) = [1 - x^{1/(n-1)} + (2ln(x) - 1)/(2n)]/(nx)$, so f' has the same sign as $g(x) := 1 - x^{1/(n-1)} + (2ln(x) - 1)/(2n)$ for $x \in (0,1]$. We prove that g has a maximum $x^* \in (0,1]$ with $g(x^*) \le 0$. This implies that both g and f' are nonpositive. Indeed,

$$g'(x) = \frac{1}{nx} - \frac{x^{\frac{1}{n-1}-1}}{n-1},$$

$$g''(x) = -\frac{1}{nx^2} + \frac{n-2}{(n-1)^2}x^{\frac{1}{n-1}-2}.$$

Thus, $g'(x^*) = 0$ only when $x^* = [(n-1)/n]^{n-1}$. Furthermore, g'' has the same sign as $h(x) := -1/n + [n-2]/[(n-1)^2]x^{1/(n-1)}$, which is an increasing function in x for all $n \ge 2$. As $h(1) = -1/[n(n-1)^2] < 0$, g'' is negative. Thus, g is concave and attains its maximum at x^*. Finally,

$$g(x^*) = \frac{1}{2n} + \frac{n-1}{n}\ln\left(1 - \frac{1}{n}\right) \le \frac{1}{2n} + \frac{n-1}{n}\left(-\frac{1}{n}\right) = \frac{1}{n^2} - \frac{1}{2n} \le 0,$$

where the last inequality follows from $n \ge 2$.   □

Endnotes

[1] It is worth mentioning that the sequence of thresholds that they found is anonymous in that they do not depend on the distribution of the random variable considered at that stage.

[2] The prophet secretary problem of our nonadaptive setting has been studied widely in the computer science community, where the approximation ratio is usually a number smaller than 1. On the other hand, our adaptive results stem from the line of research into prophet inequalities in the probability theory community. There, the ratio is usually expressed as a number larger than 1. We shall stay in line with previous literature by adhering to these conventions.

[3] Because of the choice of π_i, we actually prove the slightly stronger bound, where we maximize over $z_i \le 2q_i/[2 - (e - 2)q_i]$.

[4] The choice of π_i suggests that the random variables are not picked deterministically, but with probability less than 1, since $\pi_i < z_i^*$ if $z_i^* > 0$. However, as noted in the beginning of the proof, because of the linearity of the objective in each variable, there is always an extreme optimal solution where the random variables are picked deterministically.

[5] More precisely, the distribution has density exactly proportional to $(n-1)(1-q)^{n-2}$ in the interval A_i. Naturally, as the intervals A_i contain larger values as i progresses, the sampled quantiles will increase and the thresholds decrease.

[6] All derivatives of y are with respect to t.

[7] One may think here that the right benchmark should be the expectation of the maximum valuation. However, this cannot yield useful results. Consider a single customer whose valuation lies in $[1, +\infty)$ distributed according to $F(v) = 1 - 1/v$. Clearly, if we charge price p, then the acceptance probability is $1/p$, for a total revenue of 1. On the other hand, the expectation of the valuation is actually $+\infty$. This example can easily be turned into one with finite expectation but arbitrarily large ratio between the optimal pricing and the expectation of the random variable.

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