Equilibrium Dynamics in Market Games with Exchangeable and Divisible Resources*

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Abstract

We study a market game with $n \geq 2$ players competing over $m \geq 1$ divisible resources of different finite capacities. Resources are traded via the proportional sharing mechanism, where players are price-anticipating, meaning that they can influence the prices with their bids. Additionally, each player has an initial endowment of the resources which are sold at market prices. Although the players' total profit functions may be discontinuous in the bids, we prove existence and uniqueness of pure Nash equilibria of the resulting market game. Then, we study a discrete dynamic arising from repeatedly taking the (unique) equilibrium resource allocation as initial endowments for the next market game. We prove that the total utility value of the dynamic converges to either an optimal allocation value (maximizing total utility over the allocation space) or to a restricted optimal allocation value, where the restriction is defined by fixing some tight resources which are exclusively allocated to a single player. As a corollary, it follows that for strictly concave utility functions, the aggregated allocation vector of the dynamic converges to the unique (possibly restricted) optimal aggregated allocation, and for linear utility functions, we even get convergence of the dynamic to a (possibly restricted) optimal solution in the (non-aggregated) original allocation space.

1 Introduction

Starting with the works of Cournot [8] and Walras [25], the trading of goods using price-based market mechanisms has been a cornerstone of economic theory. A prominent model for the allocation of divisible goods via proportional sharing called the *trading post game*, has been proposed by Shapley and Shubik [24]. Each trader places a monetary bid on each good and receives each good at a fraction which equals the fraction of the own bid and the sum-aggregated bids of all players for the good. This mechanism has become the source of decentralized price-based mechanisms for allocating divisible goods with applications in telecommunications [17, 18] and cloud computing [12].

An important distinction of solution concepts for the resulting market game is whether or not the players are price-taking or price-anticipating. While the former leads to classical large-market theories à la Arrow and Debreau [1], the later renders the game to an oligopolistic model where players strategically influence the market price by their own bids. As we allow arbitrary initial endowments of resource shares, indeed, this effect may be significant as raising the own bid increases the resulting revenue for selling parts of the endowment. From a theoretical point of view, the price-anticipating mechanism is known to be considerably harder to analyze, partly because of the resulting discontinuity of the overall payoff functions.[1]

In this paper, we study the price-anticipating mechanism for a market model with exchangeable goods, assuming that the players hold initial endowments of the goods. More precisely, agents, who have quasi-linear payoff functions and are price-anticipating, trade their endowments using a proportional sharing mechanism over each resource [24, 17, 15]. In this mechanism, each agent makes a bid for each resource. Then the price of each resource is set to be the sum of the bids and each user is assigned a fraction of this resource equal to her bid divided by the resulting price. Because agents are price-anticipating, they maximize the resulting utility of their allocation minus their bids plus their revenue from selling the initial resource endowment at market prices. While the proportional sharing mechanism has been widely studied in the literature, we develop nontrivial extensions of the existing literature to establish existence and uniqueness of equilibria.

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[1] See Johari and Tsitsiklis [15] and Feldman et al. [12] for a detailed discussion.

Then, we turn to the driving question of our paper: how do the long-term dynamics of the market behave if the game is played over consecutive rounds, where the goods bought in round k serve as the endowments of round $k+1$? This discrete dynamic describes a day-to-day evolution of equilibrium market trades in terms of the changing endowments. In particular, the raised question is very similar in spirit to those asked by Brânzei, Mehta and Nisan [6] on the expanding economy, where the growth of an economy as a result of exchanging goods in rounds is studied. In their model (which originates in the growth model of von Neumann [21]), the input goods are used for producing the supply of the next round, and the properties of decentralized mechanisms leading to an expansion of the economy are studied. In our model, the "size" of the economy is fixed via assuming fixed resource capacities, but instead, we are asking how the distribution of the scarce resources changes over rounds assuming that players play equilibrium strategies in each round (while Brânzei et al. [6] use a hard-coded update formula for allocations and prices). The discrete dynamic we consider can also be cast within the area of multi-agent reinforcement learning models assuming that agents act myopically (discount factor of future utilities is high) and have full information about the other player's actions each round.

Our main result establishes that, surprisingly, the total utility value of this dynamic converges to the value of an optimal allocation.[2] Before formally describing our results, let us give a motivating example illustrating the model.

1.1 Motivating Example Suppose there are two resources $R = \{1, 2\}$ with capacities of $c_r = 1, r = 1, 2$. There are two players $N = \{1, 2\}$ with utility functions $U_1(z) = 2z$ and $U_2(z) = z$ mapping the sum-aggregated resource shares to the reals. For $n = 2$ players and $|R| = m = 2$ resources, let

$$S := \{x \in \mathbb{R}^{n\cdot m}_{\geq 0} : x_{11} + x_{21} \leq 1 = c_1, x_{12} + x_{22} \leq 1 = c_2\}$$

be the space of feasible allocations where the entries $x_{ir} \in [0, c_r], i \in N, r \in R$ denote the resource share of player i of resource r and for convenience, we write $x = (x_{11}, x_{21}; x_{12}, x_{22})$. Let $\tau \in S$ be an initial endowment of the resource shares according to $\tau_{11} := 1/4, \tau_{21} := 3/4$, $\tau_{12} := 1/2$, and $\tau_{22} := 1/2$, that is, for resource 1, player 1 holds 1/4 and player 2 holds 3/4 of the resource, and resource 2 is shared equally with a share of 1/2 each. Players submit bids $w_{ir} \geq 0$ to the market and using the proportional sharing mechanism one obtains prices $p_r, r \in R$ and resource allocations via

$$x_{ir} = \begin{cases} \frac{w_{ir}}{w_{1r}+w_{2r}}, & \text{if } w_{1r} + w_{2r} > 0 \\ 0, & \text{else,} \end{cases} \quad \text{for } i \in N, r \in R \text{ and prices } p_r = w_{1r} + w_{2r}, r \in R.$$

The payoff for player 1 of the market exchange game with the proportional sharing mechanism is given as

$$P_1(w; \tau) = U_1(x_{11} + x_{12}) - (w_{11} + w_{12}) + \tau_{11}p_1 + \tau_{12}p_2,$$

where x and p should be understood as functions of the bid vector w as explained above. The first term $U_1(x_{11} + x_{12})$ models the utility gained from acquiring a share of $x_{11} + x_{12}$ of the resources, the second term $-(w_{11} + w_{12})$ represents the payment (and hence has its own worth) and the last term $\tau_{11}p_1 + \tau_{12}p_2$ represents the revenue from selling the endowments at market prices. The payoff for player 2 is given analogously by

$$P_2(w; \tau) = U_2(x_{21} + x_{22}) - (w_{21} + w_{22}) + \tau_{21}p_1 + \tau_{22}p_2.$$

For the case $x_{ir} > 0, i \in N, r \in R$, the first order conditions

$$U_i'\left(\sum_{r\in R} x_{ir}\right)(c_r - x_{ir}) = \left(1 - \frac{\tau_{ir}}{c_r}\right)p_r$$

completely characterize the unique equilibrium allocation which is given by

$$x^1 = (x^1_{11}, x^1_{21}; x^1_{12}, x^1_{22}) = (2/5, 3/5; 2/3, 1/3).$$

[2]Except in degenerate cases that are very easy to identify. In this case the value converges to a restricted optimal value that depends on the initial endowment allocation of the goods.

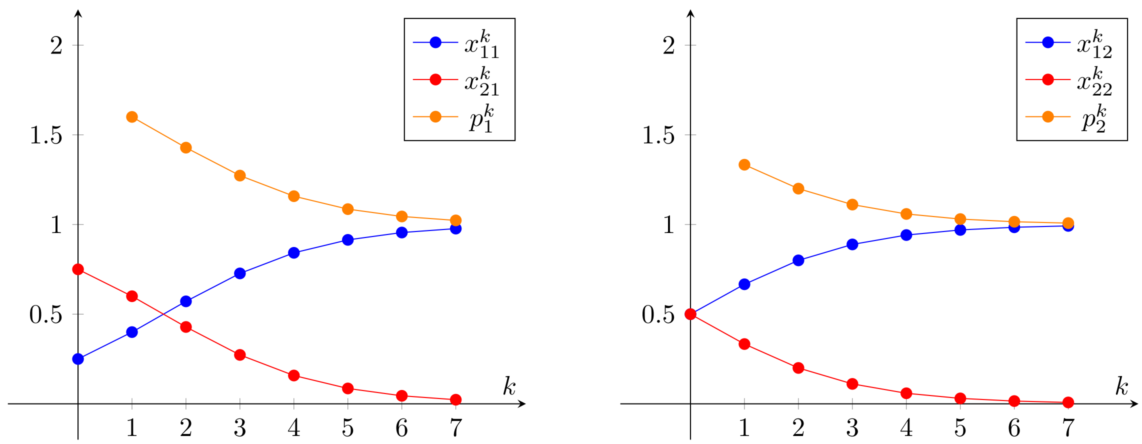

Figure 1: Illustration of the evolution of the resource shares on the resources $r = 1, 2$ for the two players $i = 1, 2$ with respect to the dynamic $x^{k+1} := f(x^k)$. The initial endowment was set to $\tau := (1/4, 3/4; 1/2, 1/2)$. The dynamic converges to the system optimal allocation $x^* = (1, 0; 1, 0)$.

The equilibrium prices are $p_1^1 = 8/5$ and $p_2^1 = 4/3$. Now let us define a dynamic that starts with $x^0 := \tau$ and evolves according to the law $x^{k+1} := f(x^k)$, where $f : S \to S$ is a function that maps any vector $\tau \in S$ to an equilibrium of the market game with endowments τ. For the above example, the resulting dynamic is depicted in Figure 1. One can observe that in the limit, player 1 acquires all resources leading to the profile $x^* = (1, 0; 1, 0)$ which maximizes the total utility $U_1(x_{11} + x_{12}) + U_2(x_{21} + x_{22})$ over $x \in S$.

1.2 Our Results We consider in this work the price-anticipating trading post game assuming that every player has a strictly increasing, concave, and smooth resource utility function mapping the sum-aggregated resource shares to the reals. The players may have an arbitrary initial endowment of the resources and the payoff of a player is defined as the resource utility gained from the purchased resource shares plus the money earned from selling the endowments (at market prices) minus the total money spent for buying resource shares. Our results for this resource exchange model are as follows.

1. We first establish equilibrium existence for arbitrary endowment allocations in Section 3. As the payoff is discontinuous in the bid-space (at 0), we cannot employ standard fixed-point theorems for concave games à la Kakutani [16]. Instead we show that Reny's [22] existence result for discontinuous games using the concept of better-reply-security is applicable to our setting. The main difficulty in the proof is to show that the better-reply security condition is in fact satisfied. Let us mention here that Feldman et al. [12] already conjectured that Reny's approach could be useful for proving existence of equilibria for the price-anticipating mechanism (although they studied the linear utility case with budgets).

2. Then, in Section 4, we show that equilibria are unique in the allocation space and essentially unique in the bid and price space. Non-uniqueness of prices and bids only arises for situations in which the endowment of a player contains the whole capacity of a resource. In this case, there is a continuum of equilibrium bids and prices for that resource, notwithstanding uniqueness of prices for other resources not having this property.

3. We finally turn to our main result in Section 5. Here we study a natural discrete dynamic of allocations, bids, and prices, by repeatedly solving the market game taking the equilibrium allocation of the previous market game as the initial endowments of the current one. This dynamic describes a day-to-day evolution of equilibrium market trades in terms of the changing endowments. We show that the total utility value of this dynamic converges to either an optimal allocation value (maximizing total utility over the allocation space) or to a restricted optimal allocation value, where the restriction is defined by fixing some tight resources which are exclusively allocated to a single player. The proof of this convergence result is non-trivial and requires several new ideas. Perhaps the most important insight is the behavior of the function f mapping initial endowments to the new equilibrium allocations (after having played the game) which we show to

be continuous.[3] This result can be interpreted as a comparative statics result describing properties of equilibria as a function of input parameters (endowments). The continuity proof for f builds on a carefully constructed sequence of allocations and prices which are analyzed using their layered tree structure. We think that this proof technique (using discrete graph-theoretic elements) can possibly be applied to show continuity (or hemi-continuity) of solution mappings (or correspondences, respectively) of more general nonlinear complementarity systems. As corollaries it follows that for strictly concave utility functions the aggregated allocation vector of the dynamic converges to the unique (possibly restricted) optimal aggregated allocation and for linear utility functions we even get convergence of the dynamic in the (non-aggregated) allocation space.

1.3 Related Work Johari and Tsitsiklis [15] studied the price-anticipating mechanism and derived an existence and uniqueness result for a single resource. Feldman et al. [12] considered the multi-resource model with linear utility functions and monetary budgets. Both works derived equilibrium existence by studying a sequence of perturbed games for which equilibria do exist (by standard fixed-point arguments) and then showed that in the limit, the resulting equilibria converge to an equilibrium of the unperturbed game.

Brânzei [4] gives an overview on proportional bid and allocation dynamics for exchange market games including the trading post game with linear utilities and player-specific budgets. In particular, Brânzei, Devanur and Rabani [5] showed that the proportional response dynamic converges to an equilibrium. Brânzei, Mehta and Nisan [6] studied a discrete dynamic for the trading post market game, where the proportional update rule for bids is implemented. As noted by Brânzei et al., for additive valuations, this dynamic can be interpreted as a gradient descent with respect to the Bregman divergence. Azad and Mussachio [2] studied the trading post game for a single resource with endowments and linear utility functions. They considered two stages of the game and solved it using subgame perfect equilibria. In Dubey and Shubik [11], agents are free to choose which quantity of each good they agree to trade. Note that in our model, at equilibrium, agents only sell their goods if that is beneficial to them. That is, if the earnings from trading are at least the utility loss - this follows from the fact that bids can always be placed such that the initial endowments are restored (note that there are no budget constraints in our model). Therefore, players can essentially hold back parts of their endowments (and they do so if and only if that benefits them). Concerning the dynamic version, where we play the game over consecutive rounds, it follows that if the aggregated resource share of a player is smaller in the limit than it was in the initial endowment, then this player is still better off in the limit since the money earned over all rounds of the game is at least as much as the utility loss.

There is a substantial body of literature for the algorithmic problem of computing market equilibria for several related versions of exchange market games, see among others Devanur et al. [9], Duan and Mehlhorn [10], Garg and Végh [13] and Zhang [26].

For a comparison of existing models for strategic market games, see for example the surveys by Giraud [14] and Levando [20].

2 Exchange Markets with Divisible Resources

Suppose we have a set of players (or users) N with $|N| = n \geq 2$, a set of resources (or goods) R with $|R| = m \geq 1$, and each resource $r \in R$ has a capacity $c_r > 0$. We denote by $x_{ir} \geq 0$ the amount of resource $r \in R$ allocated to player $i \in N$, where $\sum_{i\in N} x_{ir} \leq c_r$, and $U_i(\sum_{r\in R} x_{ir})$ describes the utility achieved by player $i \in N$. We assume that all utility functions U_i are concave and strictly increasing over the domain $[0, \infty)$, and continuously differentiable over the domain $(0, \infty)$ with $U_i'(0) := \lim_{\varepsilon\to 0} U_i'(\varepsilon) < \infty$ for all $i \in N$. Additionally, players are equipped with an initial endowment $\tau = (\tau_{ir})_{i\in N, r\in R}$ of the resources, that is, $\tau_{ir} \geq 0$ for all $i \in N, r \in R$, and $\sum_{i\in N} \tau_{ir} \leq c_r$ for all $r \in R$.

2.1 Proportional Sharing Mechanism Suppose every user $i \in N$ gives for each resource $r \in R$ a payment (or bid) $w_{ir} \geq 0$ to the market manager. The price of resource $r \in R$ is then set to $p_r := \sum_{i\in N} w_{ir}$. If $p_r = 0$ holds for resource $r \in R$, the users receive zero units $x_{ir} := 0$ for all $i \in N$ of resource r, and if $p_r > 0$, player $i \in N$ receives $x_{ir} := c_r w_{ir}/p_r$ units of resource r.

[3] Note that by our uniqueness result we can indeed speak of a function and not of a correspondence.

2.2 Price Anticipating Users We want to study a game where users strategically announce payments $w = (w_{ir})_{i\in N, r\in R}$ so as to maximize their payoff myopically. The payoff of player $i \in N$ is defined as

$$P_i(w;\tau) = U_i\left(\sum_{r\in R} x_{ir}\right) - \sum_{r\in R} w_{ir} + \sum_{r\in R} p_r \frac{\tau_{ir}}{c_r}.$$

The first term is the utility of receiving $\sum_{r\in R} x_{ir}$ units of resources, where

$$x_{ir} = \begin{cases} \frac{c_r w_{ir}}{p_r}, & \text{if } p_r = \sum_{i\in N} w_{ir} > 0, \\ 0, & \text{else,} \end{cases}$$

the second term accounts for the payment, and the last term for the reimbursement from selling the initial endowments of the goods at market prices.

2.3 The System Optimum A system optimum is an allocation that maximizes the total utility gained by all players over the allocation space. That is, it can be described by the following optimization problem.

$$\begin{aligned} \text{(SYSTEM)} \quad \max \quad & \sum_{i\in N} U_i\left(\sum_{r\in R} x_{ir}\right) & \\ \text{s.t.:} \quad & \sum_{i\in N} x_{ir} = c_r & \forall r \in R, \\ & x_{ir} \geq 0 & \forall\, i \in N,\ \forall r \in R. \end{aligned}$$

Note that since the utility functions are strictly increasing, we have that all resources are fully allocated in any system optimum.

Since the feasible region forms a nonempty polytope and the objective function is continuous, we get existence of an optimal solution by the theorem of Weierstrass. Moreover, due to the concavity of the objective function, the optimal solutions are characterized by the necessary and sufficient KKT conditions which can be stated as follows. An allocation $x \geq 0$ with $\sum_{i\in N} x_{ir} = c_r$ for all $r \in R$ is optimal if and only if there exist $\mu_r \geq 0$ for all $r \in R$ such that

$$U_i'\left(\sum_{r\in R} x_{ir}\right) \begin{cases} = \mu_r, & \text{if } x_{ir} > 0, \\ \leq \mu_r, & \text{if } x_{ir} = 0 \end{cases}$$

holds for all $i \in N$ and $r \in R$.

Using that for any $r \in R$ there exists $i \in N$ with $x_{ir} > 0$, we get $\mu_r = \mu := \max_{i\in N} U_i'\left(\sum_{r\in R} x_{ir}\right)$ for all $r \in R$. Consequently, an allocation $x \geq 0$ with $\sum_{i\in N} x_{ir} = c_r$ for all $r \in R$ is optimal if and only if $x_{ir} > 0$ implies $U_i'\left(\sum_{r\in R} x_{ir}\right) = \max_{j\in N} U_j'\left(\sum_{r\in R} x_{jr}\right)$.

3 Existence of Nash Equilibria

A bid vector w is a pure Nash equilibrium if users maximize their payoff. We show in this section that pure Nash equilibria are guaranteed to exist in our model.

Given payments w_{-i} of the other players, player i's optimization problem can be stated as follows, where $p_r = \sum_{j\in N} w_{jr}$.

$$\begin{aligned} \max \quad & U_i\left(\sum_{r\in R:\ w_{ir}>0} \frac{c_r w_{ir}}{p_r}\right) - \sum_{r\in R} w_{ir} + \sum_{r\in R} p_r \frac{\tau_{ir}}{c_r} \\ \text{s.t.:} \quad & w_{ir} \geq 0 \quad \forall r \in R. \end{aligned}$$

Using that $p_r = \sum_{j\in N} w_{jr}$, the objective function can be reformulated as

$$U_i\left(\sum_{r\in R:\ w_{ir}>0} c_r\left(1 - \frac{\sum_{j\neq i} w_{jr}}{\sum_{j\neq i} w_{jr} + w_{ir}}\right)\right) - \sum_{r\in R} w_{ir}(1 - \frac{\tau_{ir}}{c_r}) + \sum_{r\in R}\sum_{j\in N\setminus i} w_{jr}\frac{\tau_{ir}}{c_r}.$$

Assume now that w is an equilibrium and consider an arbitrary player $i \in N$. Define $R^0 := \{r \in R : \sum_{j\neq i} w_{jr} = 0\}$ and $R^+ := \{r \in R : \sum_{j\neq i} w_{jr} > 0\}$. Observe that $\tau_{ir} = c_r$ holds for all $r \in R^0$, and $\tau_{ir} < c_r$ holds for all $r \in R^+$, since $w_i = (w_{ir})_{r\in R}$ is an optimal solution for player i's optimization problem: If $\sum_{j\neq i} w_{jr} = 0$ and $\tau_{ir} < c_r$ for some resource $r \in R$, then there is no optimal solution (consider $w_{ir} \to 0$, but $w_{ir} > 0$); if $\sum_{j\neq i} w_{jr} > 0$ and $\tau_{ir} = c_r$, we again conclude that there is no optimal solution (payment and reimbursement for r offset each other and $w_{ir} \to \infty$ leads to $x_{ir} \to c_r$, but always $x_{ir} < c_r$).

Note further that for any resource $r \in R^0$, player i may choose any positive bid $w_{ir} > 0$ (since $\tau_{ir} = c_r$). Therefore we may restrict our attention to resources $r \in R^+$, where the vector $(w_{ir})_{r\in R^+}$ is an optimal solution for the following optimization problem.

$$\begin{aligned} \max \quad & U_i\left(\sum_{r\in R^+} c_r\left(1 - \frac{\sum_{j\neq i} w_{jr}}{\sum_{j\neq i} w_{jr} + w_{ir}}\right) + \sum_{r\in R^0} c_r\right) - \sum_{r\in R^+} w_{ir}(1 - \frac{\tau_{ir}}{c_r}) \\ \text{s.t.:} \quad & w_{ir} \geq 0 \quad \forall r \in R^+. \end{aligned}$$

Furthermore, the objective of this problem is strictly concave (note that the first term is a composition of U_i, which is concave and strictly increasing, and a strictly concave function; with this, one can easily verify the definition of strict concavity). Thus the unique optimal solution $(w_{ir})_{r\in R^+}$ is characterized by the following KKT conditions.

$$\begin{aligned} U_i'\left(\sum_{r\in R^+} c_r\left(1 - \frac{\sum_{j\neq i} w_{jr}}{\sum_{j\neq i} w_{jr} + w_{ir}}\right) + \sum_{r\in R^0} c_r\right) \frac{c_r \sum_{j\neq i} w_{jr}}{(\sum_{j\neq i} w_{jr} + w_{ir})^2} - 1 + \frac{\tau_{ir}}{c_r} + \bar{\nu}_{ir} &= 0 \ \forall r \in R^+ \\ \bar{\nu}_{ir}\, w_{ir} &= 0 \ \forall r \in R^+ \\ \bar{\nu}_{ir} &\geq 0 \ \forall r \in R^+ \\ w_{ir} &\geq 0 \ \forall r \in R^+. \end{aligned}$$

Using $p_r = \sum_{j\in N} w_{jr} > 0$ and $x_{ir} = \frac{c_r w_{ir}}{p_r}$ for all $r \in R^+$, the stationarity conditions can equivalently be stated as

$$\begin{aligned} & U_i'\left(\sum_{r\in R} x_{ir}\right) \frac{c_r - x_{ir}}{p_r} - 1 + \frac{\tau_{ir}}{c_r} + \bar{\nu}_{ir} = 0 \ \forall r \in R^+ \\ \Leftrightarrow\ & U_i'\left(\sum_{r\in R} x_{ir}\right)(c_r - x_{ir}) = \left(1 - \frac{\tau_{ir}}{c_r} - \bar{\nu}_{ir}\right) p_r \ \forall r \in R^+. \end{aligned}$$

We have thus derived the following necessary equilibrium conditions.

Proposition 3.1. (Necessary Equilibrium Conditions) *If w is a pure Nash equilibrium with induced allocation x and prices p, the following conditions hold.*

1. *For $r \in R$ such that $\tau_{ir} = c_r$ for some $i \in N$, we get $w_{ir} > 0$ and $w_{jr} = 0$ for all $j \neq i$. In particular, $x_{ir} = 1$ and $x_{jr} = 0$ for all $j \neq i$.*
2. *For $r \in R' := \{r \in R : \tau_{ir} < c_r \text{ for all } i \in N\}$, there are at least two positive bids for resource r, that is, $|\operatorname{supp}\{w_r\}| \geq 2$. Furthermore, for all $i \in N$ and $r \in R'$, there exists $\bar{\nu}_{ir} \geq 0$ such that*

$$\begin{aligned} U_i'\left(\sum_{r\in R} c_r\left(1 - \frac{\sum_{j\neq i} w_{jr}}{\sum_{j\neq i} w_{jr} + w_{ir}}\right)\right) \frac{c_r \sum_{j\neq i} w_{jr}}{(\sum_{j\neq i} w_{jr} + w_{ir})^2} - 1 + \frac{\tau_{ir}}{c_r} + \bar{\nu}_{ir} &= 0 \\ \bar{\nu}_{ir}\, w_{ir} &= 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} U_i'\left(\sum_{r\in R} x_{ir}\right)(c_r - x_{ir}) &= \left(1 - \frac{\tau_{ir}}{c_r} - \bar{\nu}_{ir}\right) p_r \\ \bar{\nu}_{ir}\, x_{ir} &= 0. \end{aligned}$$

In the next theorem, we show existence of pure Nash equilibria. Note that due to the discontinuity of the payoff functions at $w = 0$, standard approaches for showing existence of equilibria like Rosen [23] cannot be directly applied. We instead exploit the result of Reny [22] for discontinuous games, which is for instance also used in [7] for a game-theoretic Fisher market model.

Let us further point out that the work by Shapley and Shubik [24] already contains an existence result for a model with slight differences but very similar to ours. However, for completeness, we think it is good to have a proof of existence, since Shapley and Shubik omit the proof and say "The details of the proof, which are rather lengthy, will be given elsewhere." On the other hand, a proof of existence for this model does not appear to be written down anywhere, and other papers like Feldman et al. [12] also do not use Shapley and Shubik's existence result.

THEOREM 3.1. *There exists a pure Nash equilibrium.*

Proof. First consider the case that $\tau_{ir} < c_r$ for all $i \in N$ and all $r \in R$. In that case, we prove existence by using the result of Reny [22] for discontinuous games. It states that a compact, quasiconcave, and better-reply secure game possesses a pure Nash equilibrium, where

- *compact* means that the strategy spaces are compact (and nonempty) and the profit functions are bounded;
- *quasiconcave* means that the strategy spaces are convex and the players' payoff functions are quasiconcave in their own strategies (for any fixed strategies of the other players);
- *better-reply secure* means that in a non-equilibrium profile, there exists a player who can secure a strictly larger payoff even if the other players deviate slightly (formal details will be given below).

Note first that we may restrict each player i's strategy space to a nonempty polytope, namely for each player $i \in N$ and resource $r \in R$ we may add the upper bound $w_{ir} \leq \frac{U_i(\sum_{r \in R} c_r) - U_i(0)}{1 - \tau_{ir}/c_r}$ since any larger choice of w_{ir} leads to a strictly smaller profit than choosing all bids equal to zero. Clearly, in this restricted (but equivalent) game, all profit functions are bounded.

To see that the payoff functions are quasiconcave, consider a player $i \in N$ and fixed strategies w_{-i} of the other players. With $R^+ := \{r \in R : \sum_{j \neq i} w_{jr} > 0\}$, we get that

$$\sum_{r \in R} x_{ir} = \sum_{r \in R^+} c_r \left(1 - \frac{\sum_{j \neq i} w_{jr}}{\sum_{j \neq i} w_{jr} + w_{ir}}\right) + \sum_{r \in R \setminus R^+ : w_{ir} > 0} c_r.$$

Note that this constitutes a concave function (even strictly concave if $R^+ \neq \emptyset$). Since U_i is strictly increasing and concave, we get that $U_i(\sum_{r \in R} x_{ir})$ is concave. Altogether, player i's profit function

$$\begin{aligned} P_i(w; \tau) &= U_i\left(\sum_{r \in R} x_{ir}\right) - \sum_{r \in R} w_{ir} + \sum_{r \in R} p_r \frac{\tau_{ir}}{c_r} \\ &= U_i\left(\sum_{r \in R} x_{ir}\right) - \sum_{r \in R} \left(1 - \frac{\tau_{ir}}{c_r}\right) w_{ir} + \sum_{r \in R} \sum_{j \in N \setminus \{i\}} w_{jr} \frac{\tau_{ir}}{c_r} \end{aligned}$$

is concave in w_i.

It remains to show better-reply security. To this end, let w be a non-equilibrium strategy profile and consider a sequence of strategy profiles w^n converging to w such that the payoffs also converge, and define $P_i := \lim_{n \to \infty} P_i(w^n; \tau)$. By definition of better-reply security, we need to show that there is a player i, a strategy $\bar{w}_i$ for this player, and a value $\alpha > P_i$ such that $P_i(\bar{w}_i, w'_{-i}; \tau) \geq \alpha$ for all w'_{-i} in some open neighbourhood of w_{-i}. Note that to this end, it suffices to find a player i and a strategy $\bar{w}_i$ such that P_i is continuous at $(\bar{w}_i, w_{-i})$ and $P_i(\bar{w}_i, w_{-i}; \tau) > P_i$. We consider two different cases. Note that the players' profit functions are continuous at w if and only if $\sum_{i \in N} w_{ir} > 0$ for all $r \in R$. Assume first that $\sum_{i \in N} w_{ir} > 0$ for all $r \in R$. In that case, all players' profit functions are continuous at w, showing that $P_i = P_i(w; \tau)$ for all $i \in N$. Since w is no equilibrium, there exists a player i and a strategy $\bar{w}_i$ with $P_i(\bar{w}_i, w_{-i}; \tau) > P_i$. If P_i is continuous at $(\bar{w}_i, w_{-i})$, which is equivalent to $\sum_{j\neq i} w_{jr} + \bar{w}_{ir} > 0$ for all $r \in R$, we are done. In particular, if $\sum_{j\neq i} w_{jr} > 0$ for all $r \in R$ we are done, so we may now assume that $\sum_{j\neq i} w_{jr} = 0$ for some $r \in R$. Define $R^0 \neq \emptyset$ as the set of such resources, and note that $w_{ir} > 0$ for any $r \in R^0$ since we assumed that $\sum_{i\in N} w_{ir} > 0$ for all $r \in R$. But then we may just choose $\bar{w}_{ir} := w_{ir}/2 > 0$ for all $r \in R^0$ and $\bar{w}_{ir} := w_{ir}$ for all $r \in R \setminus R^0$ and get that P_i is continuous at $(\bar{w}_i, w_{-i})$, as well as $P_i(\bar{w}_i, w_{-i}; \tau) > P_i$, and are done. It remains to analyze the case that $\sum_{i\in N} w_{ir} = 0$ for some $r \in R$. Define $R' \neq \emptyset$ as the set of such resources, and $R^+ := R \setminus R'$. Note that for all $i \in N$, the limit P_i of payoffs is at most

$$P_i^* := U_i\left(\sum_{r\in R^+} x_{ir} + \sum_{r\in R'} c_r\right) - \sum_{r\in R}\left(1 - \frac{\tau_{ir}}{c_r}\right) w_{ir} + \sum_{r\in R}\sum_{j\in N\setminus\{i\}} w_{jr}\frac{\tau_{ir}}{c_r},$$

and there exists a player i with $P_i < P_i^*$. For such a player i, choose $0 < \varepsilon < \frac{P_i^* - P_i}{\sum_{r\in R'}(1-\tau_{ir}/c_r)}$ and define $w_{ir} := \varepsilon$ for all $r \in R'$, $w_{ir} := w_{ir}$ for all $r \in R^+$. This yields

$$P_i(w_i, w_{-i}) - P_i = P_i^* - P_i - \varepsilon \sum_{r\in R'} (1 - \frac{\tau_{ir}}{c_r}) > 0.$$

Since P_i is continuous at (w_i, w_{-i}), we are done also in this case.

Altogether, by using the result of Reny, we get that pure Nash equilibria exist if $\tau_{ir} < c_r$ for all $i \in N$ and all $r \in R$.

It remains to analyze the case that $\tau_{ir} = c_r$ for some $r \in R$ and $i \in N$. Define $R_i := \{r \in R : \tau_{ir} = c_r\}$ for all $i \in N$ and $R'' = \bigcup_i R_i$, and $\bar{R} := R \setminus R''$. By the above, there exists an equilibrium $\bar{w}$ for the game with resource set $\bar{R}$ and utility functions $\bar{U}_i(t) := U_i(t + \sum_{r\in R_i} c_r)$. We claim that if we complement these strategies, for $r \in R''$, by $\bar{w}_{ir} := c_r \max_{j\neq i} U_j'(0)$ if $\tau_{ir} = c_r$, and $\bar{w}_{jr} := 0$ for $j \neq i$, we get a Nash equilibrium for the original game. We need to show that $\bar{w}_i$ is a best response for any player i. Assume, by contradiction, that there is a strategy $w_i \neq \bar{w}_i$ with a higher profit for player i. We will now step by step change that strategy without decreasing player i's profit until it matches $\bar{w}_i$; contradiction. Consider first the resources $r \in R'' \setminus R_i$. If $w_{ir} \neq \bar{w}_{ir}$, that is, $\bar{w}_{ir} = 0 < w_{ir}$, we claim that setting $w_{ir} := 0 = \bar{w}_{ir}$ will not decrease player i's profit: Noting that $\tau_{ir} = 0$, the difference in profit is

$$\Delta(w_{ir}) := U_i(x_i) - U_i\left(x_i + \frac{c_r w_{ir}}{c_r \max_{j\neq i'} U_j'(0) + w_{ir}}\right) + w_{ir}$$

where x_i is the total amount with respect to w_i that player i gets from resources different from r, and i' is the player with $\tau_{i'r} = c_r$. We need to show that $\Delta(w_{ir}) \geq 0$. Consider Δ as a function in w_{ir}. Clearly, $\Delta(0) = 0$. Furthermore,

$$\begin{aligned}\Delta'(w_{ir}) &= -U_i'\left(x_i + \frac{c_r w_{ir}}{c_r \max_{j\neq i'} U_j'(0) + w_{ir}}\right) \cdot \frac{c_r^2 \max_{j\neq i'} U_j'(0)}{(c_r \max_{j\neq i'} U_j'(0) + w_{ir})^2} + 1 \\ &\geq -\frac{U_i'(0)}{\max_{j\neq i'} U_j'(0)} + 1 \geq 0\end{aligned}$$

for all $w_{ir} \geq 0$, showing the claim. Next, consider the resources in R_i, that is, with $\tau_{ir} = c_r$. If $w_{ir} \neq \bar{w}_{ir}$, we again get that setting $w_{ir} := \bar{w}_{ir} = c_r \max_{j\neq i} U_j'(0) > 0$ does not decrease player i's profit (since $\tau_{ir} = c_r$ and $\bar{w}_{-ir} = 0$, payment and reimbursement for r offset each other, and any positive bid implies that player i gets r completely). It remains to consider the resources in $\bar{R}$. But since $w_{ir} = \bar{w}_{ir}$ for all $r \in R''$ and $(\bar{w}_{ir})_{r\in\bar{R}}$ is a best response for player i in the game with resources $\bar{R}$ and utilities $\bar{U}_i$, we conclude that setting $w_{ir} = \bar{w}_{ir}$ for all $r \in \bar{R}$ can only increase player i's profit. This shows existence of a Nash equilibrium also for the case that $\tau_{ir} = c_r$ for some $r \in R$ and $i \in N$ and completes the proof. □

4 Uniqueness of Equilibria

In this section we show that there is a unique (pure Nash) equilibrium allocation. Note that to prove this, we may restrict ourselves to the case that $\tau_{ir} < c_r$ for all $i \in N, r \in R$. The reason is that by the necessary equilibrium conditions, any equilibrium allocation x consists of an equilibrium allocation for the game with resource set $\bar{R} := \{r \in R : \tau_{ir} < c_r \text{ for all } i \in N\}$ and utility functions $\bar{U}_i(t) := U_i(t + \sum_{r\in R_i} c_r)$, where $R_i := \{r \in R : \tau_{ir} = c_r\}$ for all $i \in N$, together with $x_{ir} = 1$ and $x_{-ir} = 0$ for all $r \in R_i$ and $i \in N$.

We start by showing that if we fix the total amount of resources any player gets, there can be at most one corresponding equilibrium.

PROPOSITION 4.1. *For fixed values $s_i \geq 0, i \in N$, with $\sum_{i\in N} s_i = \sum_{r\in R} c_r$, there can be at most one Nash equilibrium allocation x with $\sum_{r\in R} x_{ir} = s_i$ for all $i \in N$.*

Proof. Define the following optimization problem (P) (with parameter $s_i, i \in N$).

$$\text{(P)} \qquad \begin{aligned} \max\ & \sum_{i\in N}\sum_{r\in R} U_i'(s_i)\frac{c_r}{c_r-\tau_{ir}}\left(c_r x_{ir} - \frac{x_{ir}^2}{2}\right) \\ \text{s.t.: } & \sum_{r\in R} x_{ir} = s_i \ \forall\ i \in N, \\ & \sum_{i\in N} x_{ir} = c_r \ \forall\ r \in R, \\ & x_{ir} \geq 0 \ \forall\ r \in R, i \in N. \end{aligned}$$

Due to the strict concavity of the objective, (P) has at most one optimal solution. We show that an equilibrium allocation x with $\sum_r x_{ir} = s_i$ for all $i \in N$ is optimal for (P), proving the proposition. Note that, clearly, x is feasible for (P). Furthermore, due to the necessary and sufficient optimality conditions for (P), it suffices to give $\lambda_i, i \in N$, $\lambda_r, r \in R$, and $\mu_{ir}, r \in R, i \in N$, such that the following conditions hold:

$$\begin{aligned} U_i'(s_i)\frac{c_r}{c_r-\tau_{ir}}(c_r - x_{ir}) - \lambda_i - \lambda_r + \mu_{ir} = 0 \ \forall\ r \in R, i \in N, \\ \mu_{ir}x_{ir} = 0 \ \forall\ r \in R, i \in N, \\ \mu_{ir} \geq 0 \ \forall\ r \in R, i \in N. \end{aligned}$$

For $r \in R$ and $i \in N$, this is equivalent to

$$U_i'(s_i)\,(c_r - x_{ir}) \begin{cases} = \left(1 - \frac{\tau_{ir}}{c_r}\right)(\lambda_i + \lambda_r), & \text{for } x_{ir} > 0, \\ \leq \left(1 - \frac{\tau_{ir}}{c_r}\right)(\lambda_i + \lambda_r), & \text{for } x_{ir} = 0. \end{cases}$$

Note that since x is a Nash equilibrium allocation, we get for any player $i \in N$ and $r \in R$ that

$$U_i'\left(\sum_{r\in R} x_{ir}\right)(c_r - x_{ir}) \begin{cases} = \left(1 - \frac{\tau_{ir}}{c_r}\right)p_r, & \text{for } x_{ir} > 0, \\ \leq \left(1 - \frac{\tau_{ir}}{c_r}\right)p_r, & \text{for } x_{ir} = 0. \end{cases}$$

That means we may set $\lambda_i := 0$ and $\lambda_r := p_r$ for all $i \in N$ and $r \in R$ and get the desired conditions (since $\sum_r x_{ir} = s_i$ for all $i \in N$). ∎

Using the last proposition (as well as the existence result from the last section), we can now show uniqueness of equilibria.

THEOREM 4.1. *There is a unique Nash equilibrium allocation.*

Proof. Assume, by contradiction, that there are two different Nash equilibrium allocations, denoted by x and x', respectively. Let w and w' be the corresponding strategy profiles. By Proposition 4.1, we may assume that there is a player i who gets more units of resources in total in x than in x', that is, with $\sum_{r\in R} x_{ir} > \sum_{r\in R} x'_{ir}$. Let N^+ be the set of such players. For each $i \in N^+$, there is at least one resource $r \in R$ with $x_{ir} > x'_{ir}$. Let R_i^+ be the set of resources where player $i \in N^+$ gets more in x than in x', $R_i^+ := \{r \in R : x_{ir} > x'_{ir}\}$. Observe that for any $r \in \bigcup_{i\in N^+} R_i^+$, we have that $p_r := \sum_{i\in N} w_{ir} < \sum_{i\in N} w'_{ir} = p'_r$: To see this, note that for $i \in N^+$

with $r \in R_i^+$ we get $U_i'(\sum_{\bar r \in R} x_{i\bar r}) \le U_i'(\sum_{\bar r \in R} x'_{i\bar r})$ since $\sum_{\bar r \in R} x_{i\bar r} > \sum_{\bar r \in R} x'_{i\bar r}$ and U_i is concave. Furthermore, $x_{ir} > x'_{ir}$ holds, as well as $\tau_{ir} < c_r$ and $x_{ir} < c_r$. Combining these inequalities with the KKT conditions yields

$$\begin{aligned}\left(1 - \frac{\tau_{ir}}{c_r}\right) p_r &= U_i'\left(\sum_{\bar r \in R} x_{i\bar r}\right)(c_r - x_{ir}) \\ &< U_i'\left(\sum_{\bar r \in R} x'_{i\bar r}\right)(c_r - x'_{ir}) \\ &\le \left(1 - \frac{\tau_{ir}}{c_r}\right) p'_r\end{aligned}$$

where we additionally used that $U_i'(\sum_{\bar r \in R} x'_{i\bar r}) > 0$. Therefore, $p_r < p'_r$. Using this, we can show that $x_{jr} \ge x'_{jr}$ holds for all $j \notin N^+$ (and $r \in \bigcup_{i \in N^+} R_i^+$): Assume, by contradiction, that $x_{jr} < x'_{jr}$ for some $j \notin N^+$. Using this as well as the KKT-conditions, the facts that U_j is strictly increasing and concave, $\tau_{jr} < c_r$ and $x_{jr} < c_r$, as well as $\sum_{\bar r \in R} x_{j\bar r} \le \sum_{\bar r \in R} x'_{j\bar r}$, we get

$$\begin{aligned}U_j'\left(\sum_{\bar r \in R} x'_{j\bar r}\right)(c_r - x'_{jr}) &= \left(1 - \frac{\tau_{jr}}{c_r}\right) p'_r \\ &> \left(1 - \frac{\tau_{jr}}{c_r}\right) p_r \\ &\ge U_j'\left(\sum_{\bar r \in R} x_{j\bar r}\right)(c_r - x_{jr}) \\ &> U_j'\left(\sum_{\bar r \in R} x'_{j\bar r}\right)(c_r - x'_{jr}),\end{aligned}$$

a contradiction. We thus have that $x_{jr} \ge x'_{jr}$ holds for all $j \notin N^+$. But this means that the players $i \in N^+$ with $r \notin R_i^+$, that is, where $x_{ir} \le x'_{ir}$, need to compensate for the increased amount of resource r that the other players get, in particular the players in N^+ with $r \in R_i^+$, that is,

$$\sum_{i \in N^+ : r \in R_i^+} (x_{ir} - x'_{ir}) \le \sum_{i \in N^+ : r \notin R_i^+} (x'_{ir} - x_{ir}).$$

Summing over all $r \in \bigcup_{i \in N^+} R_i^+$ leads to

$$\sum_{r \in \bigcup_{i \in N^+} R_i^+} \sum_{i \in N^+ : r \in R_i^+} (x_{ir} - x'_{ir}) \le \sum_{r \in \bigcup_{i \in N^+} R_i^+} \sum_{i \in N^+ : r \notin R_i^+} (x'_{ir} - x_{ir}).$$

Using this we get

$$\begin{aligned}\sum_{i \in N^+} \sum_{r \in R_i^+} (x_{ir} - x'_{ir}) &= \sum_{r \in \bigcup_{i \in N^+} R_i^+} \sum_{i \in N^+ : r \in R_i^+} (x_{ir} - x'_{ir}) \\ &\le \sum_{r \in \bigcup_{i \in N^+} R_i^+} \sum_{i \in N^+ : r \notin R_i^+} (x'_{ir} - x_{ir}) \\ &\le \sum_{i \in N^+} \sum_{r \in R \setminus R_i^+} (x'_{ir} - x_{ir}),\end{aligned}$$

where we used for the last inequality that we sum over a superset, and $x'_{ir} - x_{ir} \ge 0$ whenever $i \in N^+$ and $r \in R \setminus R_i^+$. But this now implies

$$\sum_{i \in N^+} \sum_{r \in R} x_{ir} \le \sum_{i \in N^+} \sum_{r \in R} x'_{ir},$$

contradicting the fact that the players in N^+ get more amount of resources in total in x than in x'. Therefore, there can be at most one equilibrium allocation, which proves the theorem since existence is guaranteed by Theorem 3.1. ∎

REMARK 4.1. *We showed in Theorem 4.1 that the equilibrium allocation x is unique. Regarding the equilibrium prices, note that if $\tau_{ir} < c_r$ for all $i \in N$ holds for a resource r, the equilibrium price p_r is uniquely defined since there exists a player i with $x_{ir} > 0$ and thus $p_r = \frac{U_i'(\sum_{r' \in R} x_{ir'})(c_r - x_{ir})}{1 - \frac{\tau_{ir}}{c_r}}$ by the necessary equilibrium conditions. In particular, equilibrium strategies $w_{ir} = x_{ir} p_r / c_r$ are then uniquely defined for resource r.*

However, the equilibrium price is not uniquely defined for a resource r with $\tau_{ir} = c_r$ for some player i: We know from the necessary equilibrium conditions that $w_{jr} = 0$ for all $j \neq i$. Furthermore, we showed in the proof of Theorem 3.1 that $w_{ir} := c_r \max_{j \neq i} U_j'(0)$ leads to an equilibrium. However, any larger bid for player i is then also an equilibrium strategy. This shows that the equilibrium price (which is equal to player i's bid) for r is not uniquely defined.

5 Equilibrium Dynamics

In this section, we analyze the long-term dynamic of the market if the game is played over consecutive rounds where the goods bought in round k serve as the endowments of round $k+1$. Formally, let

$$S := \{x \in \mathbb{R}^{n \cdot m}_{\geq 0} : \sum_{i \in N} x_{ir} \leq c_r \text{ for all } r \in R\}$$

be the set of feasible endowments, and $f : S \to S$ the function mapping any vector $x \in S$ to the equilibrium allocation of the market game with endowments x (note that f is well-defined since by Theorem 4.1 there is a unique equilibrium allocation). We study in this section the sequence of equilibrium allocations $(x^k)_{k \geq 0}$ defined by

$$x^0 := \tau \in S \text{ and } x^{k+1} := f(x^k) \text{ for } k \geq 0. \tag{5.1}$$

That is, we study a discrete nonlinear dynamical system defined by the function f together with an initial endowment $\tau \in S$. First of all, the following example shows that the limit of the sequence $(x^k)_{k \geq 0}$ is sensitive to the initial endowment τ.

EXAMPLE 5.1. *We consider an instance with two resources $R = \{1, 2\}$ and two players $N = \{1, 2\}$ with strictly concave utility functions $U_1(z) := 2\log(z+1)$ and $U_2(z) := \log(z+1)$, respectively. We compare convergence of the dynamic with respect to two different initial endowments $\tau^1 := (1/4, 3/4; 1/3, 2/3)$ and $\tau^2 := (1/2, 1/2; 1/2, 1/2)$, that is, $\tau^1_{11} = 1/4, \tau^1_{21} = 3/4, \tau^1_{12} = 1/3, \tau^1_{22} = 2/3$, whereas for τ^2 every player gets for every resource an initial share of $1/2$.*

From the two Figures 2 and 3 it follows that both dynamics converge to a system optimal allocation, but the limit point depends on the respective endowment.

In the above example, both sequences converge to a system optimal allocation, that is, an allocation with largest possible total utility $U^* = \max_{x \in S} U(x)$, where $U(x) := \sum_{i \in N} U_i(\sum_{r \in R} x_{ir})$. In fact, we show in the following that the total utility $U(x^k) := \sum_{i \in N} U_i(\sum_{r \in R} x^k_{ir})$ always converges to U^* if the initial endowment x^0 satisfies $x^0_{ir} < c_r$ for all $r \in R$. If $x^0_{ir} = c_r$ for some $r \in R$ and $i \in N$, then by the necessary equilibrium conditions the allocation for resource r will not change throughout the sequence of allocations, that is, $x^k_{ir} = c_r$ and $x^k_{-ir} = 0$ for all $k \geq 0$. Denote $R'(x^0) := \{r \in R : x^0_{ir} = c_r \text{ for some } i \in N\}$. For the general case where $R'(x^0) \neq \emptyset$, we can show that the total utility $U(x^k)$ converges to the best-possible value with respect to the fixed allocations given by the initial endowment,

$$U^*(x^0) := \max\{U(x) : x \in S, x_r = x^0_r \text{ for all } r \in R'(x^0)\}.$$

To this end, we first show in the next proposition that if the equilibrium allocation x (of the one-round game) equals the initial endowment (that is, x is a fixed point of f), then x is best-possible w.r.t. the fixed allocations given by the initial endowment. Furthermore, we show that at non-fixed points of f, the total utility is strictly increasing. This implies that the sequence of utilities $(U(x^k))_{k \geq 0}$ is non-decreasing and, consequently, converges (as it is obviously bounded).

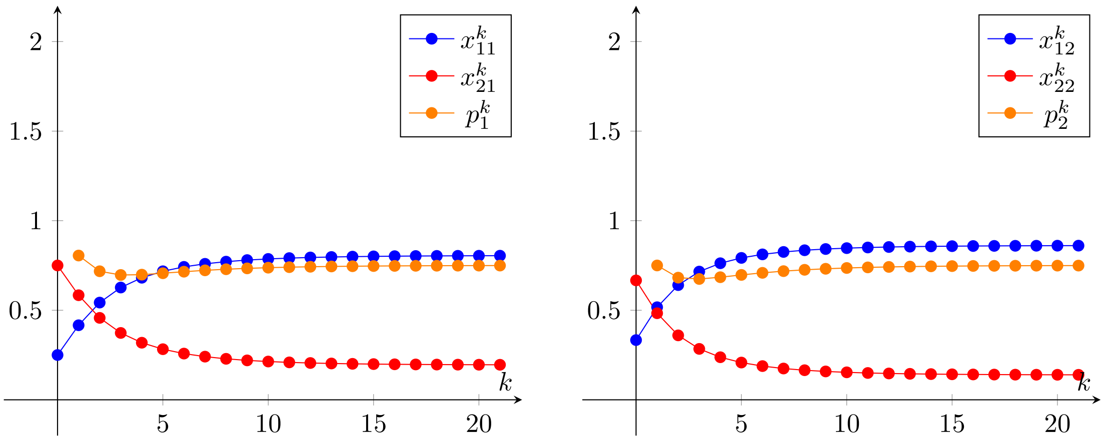

Figure 2: Illustration of the evolution of the resource shares on the resources $r = 1, 2$ for the two players $i = 1, 2$ with respect to the dynamic $x^{k+1} := f(x^k)$. The initial endowment was set to $\tau^1 := (1/4, 3/4; 1/3, 2/3)$. The dynamic converges to a system optimal allocation $x(\tau^1)$ with utility $U(x(\tau^1)) \approx 2.249$. The player-specific limits are given by $\bar{x}(\tau^1) \approx (0.805, 0.195; 0.861, 0.139)$.

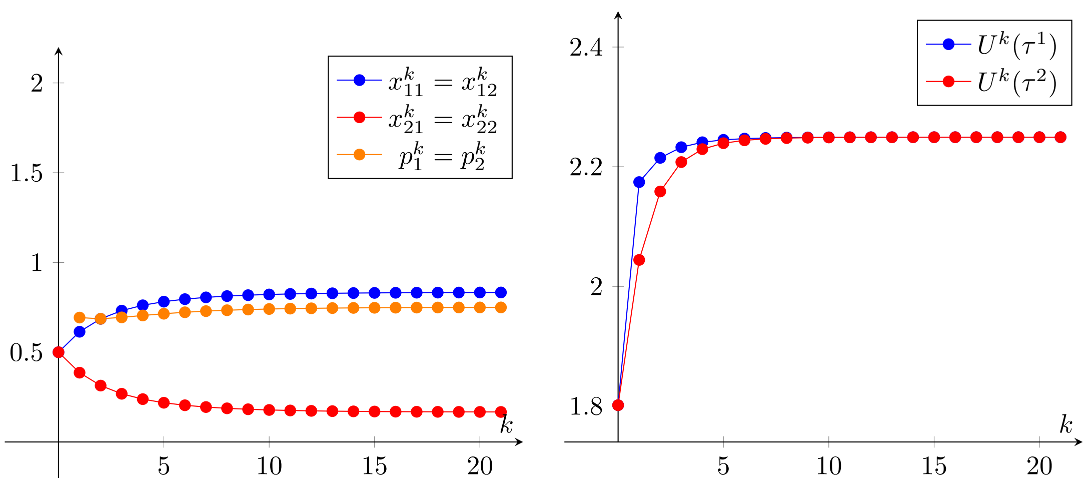

Figure 3: The initial endowment was set to $\tau^2 := (1/2, 1/2; 1/2, 1/2)$. The dynamic for both resources is completely symmetric and therefore both resource and price quantities on the two resources are identical. The player-specific limits are given by $\bar{x}(\tau^2) \approx (0.833, 0.167; 0.833, 0.167)$. The right figure compares the two sequences of utility values $U(x^k(\tau^1)) =: U^k(\tau^1)$ and $U(x^k(\tau^2)) =: U^k(\tau^2)$. Both converge to a system optimal value of $\lim_{k\to\infty} U^k(\tau^1) = \lim_{k\to\infty} U^k(\tau^2) \approx 2.249$.

PROPOSITION 5.1. *Let $x \in S$ and $R'(x) := \{r \in R : x_{ir} = c_r \text{ for some } i \in N\}$.*

1. *If $x = f(x)$ is a fixed point of f, then x is socially optimal among all allocations x' with $x'_r = x_r$ for all $r \in R'(x)$.*
2. *If $x \neq f(x)$, then $U(f(x)) > U(x)$.*

Proof. Assume first that $x = f(x)$. Since $x = f(x)$ is the equilibrium allocation in the game with initial endowment x, we have for any player $i \in N$ and any resource $r \in R \setminus R'(x)$ that

$$U_i'\left(\sum_{r\in R} x_{ir}\right)(c_r - x_{ir}) = \left(1 - \frac{x_{ir}}{c_r}\right) p_r \text{ if } x_{ir} > 0,$$
$$U_i'\left(\sum_{r\in R} x_{ir}\right)(c_r - x_{ir}) \leq \left(1 - \frac{x_{ir}}{c_r}\right) p_r \text{ if } x_{ir} = 0,$$

which is equivalent to (note that $x_{ir} < c_r$)

$$U_i'\left(\sum_{r\in R} x_{ir}\right) = \frac{p_r}{c_r} \text{ if } x_{ir} > 0,$$
$$U_i'\left(\sum_{r\in R} x_{ir}\right) \leq \frac{p_r}{c_r} \text{ if } x_{ir} = 0.$$

In other words, $x_{ir} > 0$ implies $U_i'\left(\sum_{r\in R} x_{ir}\right) = \max_{j\in N} U_j'\left(\sum_{r\in R} x_{jr}\right)$ for all $i \in N$ and $r \notin R'(x)$, which are exactly the needed optimality conditions.

Now consider the case that $x \neq f(x) =: \bar{x}$. We need the following claim.

CLAIM 5.1. *The following implications are true for all $i \in N$ and all $r \in R \setminus R'(x)$:*

$$\bar{x}_{ir} < x_{ir} \Rightarrow U_i'\left(\sum_{r\in R} \bar{x}_{ir}\right) - \frac{p_r}{c_r} < 0$$
$$\bar{x}_{ir} > x_{ir} \Rightarrow U_i'\left(\sum_{r\in R} \bar{x}_{ir}\right) - \frac{p_r}{c_r} > 0$$

Proof. [of Claim 5.1] Since $\bar{x} = f(x)$ is the Nash equilibrium allocation for the game with initial endowment x we have that

$$U_i'\left(\sum_{r\in R} \bar{x}_{ir}\right)(c_r - \bar{x}_{ir}) = \left(1 - \frac{x_{ir}}{c_r}\right) p_r \text{ if } \bar{x}_{ir} > 0,$$
$$U_i'\left(\sum_{r\in R} \bar{x}_{ir}\right)(c_r - \bar{x}_{ir}) \leq \left(1 - \frac{x_{ir}}{c_r}\right) p_r \text{ if } \bar{x}_{ir} = 0$$

for any player $i \in N$ and resource $r \in R \setminus R'(x)$. Note that since $x_{ir} < c_r$, we have $\bar{x}_{ir} < c_r$ for all $r \in R \setminus R'(x)$ and $i \in N$. Thus the Nash conditions can equivalently be stated as

$$U_i'\left(\sum_{r\in R} \bar{x}_{ir}\right) = \frac{c_r - x_{ir}}{c_r - \bar{x}_{ir}} \cdot \frac{p_r}{c_r} \text{ if } \bar{x}_{ir} > 0,$$
$$U_i'\left(\sum_{r\in R} \bar{x}_{ir}\right) \leq \frac{c_r - x_{ir}}{c_r - \bar{x}_{ir}} \cdot \frac{p_r}{c_r} \text{ if } \bar{x}_{ir} = 0.$$

Noting that $p_r > 0$ for all $r \in R \setminus R'(x)$, this leads to the stated inequalities. ∎

Using Claim 5.1 as well as $x \neq \bar{x}$, we now obtain

$$\begin{aligned}
&\sum_{i\in N} U_i'\left(\sum_{r\in R}\bar{x}_{ir}\right)\sum_{r\in R}(x_{ir}-\bar{x}_{ir}) \\
&=\sum_{i\in N}\sum_{r\in R}\left(U_i'\left(\sum_{r\in R}\bar{x}_{ir}\right)-\frac{p_r}{c_r}\right)(x_{ir}-\bar{x}_{ir})+\sum_{i\in N}\sum_{r\in R}\frac{p_r}{c_r}(x_{ir}-\bar{x}_{ir}) \\
&\leq\sum_{i\in N}\sum_{r\in R}\left(U_i'\left(\sum_{r\in R}\bar{x}_{ir}\right)-\frac{p_r}{c_r}\right)(x_{ir}-\bar{x}_{ir}) \\
&< 0.
\end{aligned}$$

Using this as well as the concavity of the utility functions U_i, $i \in N$, we get

$$\begin{aligned}
U(\bar{x})=\sum_{i\in N}U_i\left(\sum_{r\in R}\bar{x}_{ir}\right) &> \sum_{i\in N}\left(U_i\left(\sum_{r\in R}\bar{x}_{ir}\right)+U_i'\left(\sum_{r\in R}\bar{x}_{ir}\right)\sum_{r\in R}(x_{ir}-\bar{x}_{ir})\right) \\
&\geq \sum_{i\in N}U_i\left(\sum_{r\in R}x_{ir}\right)=U(x),
\end{aligned}$$

as desired. ∎

As already stated, we know from the above proposition that the sequence of utilities $(U(x^k))_{k\geq 0}$ converges. It remains to show that the limit is best-possible (w.r.t. the fixed allocations given by the initial endowment x^0). To this end, it obviously suffices to show that $U(\bar{x})$ is best-possible, where $\bar{x}$ is an accumulation point of the sequence of allocations (x^k). Therefore, we study in the following the accumulation points of (x^k), with the result that any accumulation point is a fixed point of f (see Proposition 5.3). To show this, we first prove that the equilibrium allocation (of the one-round game) is continuous w.r.t the initial endowment vector τ, that is, the function f is continuous. The proof idea is to define an optimization problem with the property that the optimal solutions describe the equilibrium allocation, and then to use Berge's theorem of the maximum. The optimization problem that we use is essentially the formulation of the equilibrium conditions as a complementarity problem, but with an additional upper bound on the price vector to make the feasible solution space compact (required for Berge's theorem).

Proposition 5.2. *The function $f : S \to S$ mapping endowments to equilibrium allocations is continuous.*

Proof. For $x \in S$, define $R^f(x) := \{r \in R : x_{ir} = c_r \text{ for some } i \in N\}$ and $R_i(x) := \{r \in R : x_{ir} = c_r\}$ for all $i \in N$. Consider the following optimization problem (EQP(x)) with variables $x'_{ir}, i \in N, r \in R$ and $p'_r, r \in R$.

(EQP(x)) $$\min \sum_{i\in N, r\in R} x'_{ir}\left(\left(1-\frac{x_{ir}}{c_r}\right)p'_r - U_i'\left(\sum_{r\in R}x'_{ir}\right)(c_r - x'_{ir})\right)$$

(5.2) s.t.: $$x'_{ir}\geq 0 \qquad \forall i\in N, r\in R,$$

(5.3) $$\sum_{i\in N}x'_{ir}=c_r \qquad \forall r\in R,$$

(5.4) $$p'_r\geq 0 \qquad \forall r\in R,$$

(5.5) $$p'_r\leq \max\{2, 3/c_r\}\cdot c_r^2\max_{i\in N}U_i'(0) \qquad \forall r\in R,$$

(5.6) $$\left(1-\frac{x_{ir}}{c_r}\right)p'_r - U_i'\left(\sum_{r\in R}x'_{ir}\right)(c_r - x'_{ir})\geq 0 \qquad \forall i\in N, r\in R.$$

Consider the equilibrium allocation $f(x)$ and a corresponding vector of equilibrium prices p, where we may assume w.l.o.g. that $p_r = c_r \max_{j\neq i} U_j'(0)$ for any $i \in N$ and $r \in R_i(x)$ (note that with this, p is uniquely defined, see Remark 4.1). We get the following connection between optimal solutions of EQP(x) and Nash equilibria.

CLAIM 5.2. • *$(x', p') := (f(x), p)$ is an optimal solution of (EQP(x)).*

• *Any optimal solution (x', p') of (EQP(x)) fulfills $x' = f(x)$ and $p'_r = p_r$ for all $r \in R \setminus R^f(x)$.*

Proof. [of Claim 5.2] We will first show that $(x', p') := (f(x), p)$ is an optimal solution of (EQP(x)): Clearly, (5.2) - (5.4) are fulfilled. For (5.5), consider $r \in R$ with $x_{ir} < c_r$ for all $i \in N$. Since there exists a player $j \in N$ with $x'_{jr} > 0$ and $x_{jr} \leq c_r/2$, we get from the Nash conditions that

$$p'_r = U'_j\left(\sum_{r \in R} x'_{jr}\right) \frac{c_r - x'_{jr}}{c_r - x_{jr}} \cdot c_r < 2c_r U'_j(0) \leq 2c_r \max_{i \in N} U'_i(0) < \max\{2, 3/c_r\} \cdot c_r^2 \max_{i \in N} U'_i(0).$$

For $i \in N$ and $r \in R_i(x)$ we have

$$p'_r = c_r \max_{j \neq i} U'_j(0) \leq c_r \max_{i \in N} U'_i(0) < \max\{2, 3/c_r\} \cdot c_r^2 \max_{i \in N} U'_i(0).$$

It remains to show that (5.6) is fulfilled. Consider first $r \in R^f(x)$, say $r \in R_i(x)$. Since $x_{ir} = x'_{ir} = c_r$ and $x_{jr} = x'_{jr} = 0$ for all $j \neq i$, we get that

$$\left(1 - \frac{x_{ir}}{c_r}\right) p'_r - U'_i\left(\sum_{r \in R} x'_{ir}\right)(c_r - x'_{ir}) = 0$$

and for $j \neq i$ we get

$$\begin{aligned}\left(1 - \frac{x_{jr}}{c_r}\right) p'_r - U'_j\left(\sum_{r \in R} x'_{jr}\right)(c_r - x'_{jr}) &= p'_r - U'_j\left(\sum_{r \in R} x'_{jr}\right) c_r \\ &= c_r \max_{j \neq i} U'_j(0) - U'_j\left(\sum_{r \in R} x'_{jr}\right) c_r \\ &\geq c_r \max_{j \neq i} U'_j(0) - U'_j(0) c_r \geq 0.\end{aligned}$$

For $r \in R \setminus R^f(x)$, inequality (5.6) follows from the Nash conditions. We thus know that $(x', p') = (f(x), p)$ is feasible for (EQP(x)). For optimality, note that due to (5.2) and (5.6), an objective value of zero implies optimality. Furthermore, we have that

$$x'_{ir}\left(\left(1 - \frac{x_{ir}}{c_r}\right) p'_r - U'_i\left(\sum_{r \in R} x'_{ir}\right)(c_r - x'_{ir})\right) = 0 \tag{5.7}$$

for all $i \in N$ and $r \in R$, showing optimality of (x', p'): For $r \in R \setminus R^f(x)$, we get this from the Nash conditions, and for $r \in R^f(x)$ it follows from the argumentation above (where we showed (5.6) for $r \in R^f(x)$). We have thus shown that $(x', p') = (f(x), p)$ is an optimal solution of (EQP(x)).

Conversely, any optimal solution (x', p') of (EQP(x)) fulfills $x' = f(x)$ and $p'_r = p_r$ for all $r \in R \setminus R^f(x)$. To see this, note that for $r \in R^f(x)$, say $r \in R_i(x)$, we get from $x_{ir} = c_r$ and $x'_{ir} \leq c_r$ and (5.6), that $x'_{ir} = c_r = f(x)_{ir}$ and $x'_{jr} = 0 = f(x)_{jr}$ for all $j \neq i$. For $r \in R \setminus R^f(x)$, note that we know from the first part of the proof that an optimal solution has an objective value of zero, and thus (5.7) needs to hold for all $i \in N$. This implies in particular that $x'_{ir} < c_r$ for all $i \in N$, since if $x'_{ir} = c_r > 0$ for some $i \in N$, we get from (5.7) that $p'_r = 0$, but also

$$p'_r \geq \frac{U'_j(\sum_{r \in R} x'_{jr}) c_r}{1 - \frac{x_{jr}}{c_r}} > 0$$

from (5.6) for $j \neq i$. Thus, there are at least two players i with $x'_{ir} > 0$, and $p'_r > 0$. But then, (5.6), together with (5.7) for all $r \in R \setminus R^f(x)$ and all $i \in N$, are exactly the equilibrium conditions, showing $x'_r = f(x)_r$ and $p'_r = p_r$ for all $r \in R \setminus R^f(x)$. This shows Claim 5.2. ∎

We want to use Berge's theorem of the maximum [3] to show continuity of f. To this end, note that the objective function of (EQP(x)) is continuous (as a function of x, x' and p'). For $x \in S$, let $\gamma(x)$ denote the feasible region of (EQP(x)). Then, $\gamma(x)$ is nonempty and compact. If, additionally, the correspondence γ is continuous (that is, both upper and lower hemicontinuous), we get from Berge's theorem of the maximum that the correspondence of optimal solutions, denoted by $\gamma^*(x)$, is upper-hemicontinuous. From that, we can deduce the continuity of f, using that the optimal allocation $x' = f(x)$ is unique: Let $x \in S$ and $\varepsilon > 0$. From the upper-hemicontinuity of γ^*, we get that for all open sets V with $\gamma^*(x) \subseteq V$, there exists an open set U with $x \in U$ and $\gamma^*(x') \subseteq V$ for all $x' \in U$. Thus we may just complement $B_\varepsilon(f(x))$ arbitrarily to an open superset for $\gamma^*(x)$ and get that there exists an open set U with $x \in U$ such that $f(x') \in B_\varepsilon(f(x))$ for all $x' \in U$. It thus remains to show that the correspondence γ is continuous.

Claim 5.3. *The correspondence γ is continuous.*

Proof. [of Claim 5.3] Since γ has the closed graph property and $\gamma(x)$ is contained in the polytope defined by (5.2)-(5.5), we have that γ is upper-hemicontinuous. It remains to show lower-hemicontinuity. We use the sequential characterization. Thus consider a sequence (x^n) converging to $x \in S$ with $x_n \in S$ for all $n \in \mathbb{N}$, and a feasible solution $(x', p') \in \gamma(x)$. We need to show that there exists a sequence $(\bar{x}^n, \bar{p}^n)$ that converges to (x', p'), and some $N \in \mathbb{N}$ with $(\bar{x}^n, \bar{p}^n) \in \gamma(x^n)$ for all $n \geq N$. Our idea for constructing such a sequence is to use the allocation $\bar{x}^n = x'$ and only adapt p' to achieve a solution which lies in $\gamma(x^n)$ and converges to (x', p'). However, it turns out that we might need to change the allocation slightly, too. We now describe our approach in detail. Note that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have for $r \in R \setminus R^f(x)$ that $x^n_{ir} < c_r$ for all $i \in N$, and for $i \in N$ with $r \in R_i(x)$ that $x^n_{jr} < c_r$ for all $j \neq i$. For $n \geq N$ we define $\bar{x}^n := x'$ and for $r \in R \setminus R^f(x)$ we set

$$\bar{p}^n_r := \max\left\{p'_r, \max_{i\in N} \frac{U'_i\left(\sum_{r\in R} \bar{x}^n_{ir}\right)(c_r - \bar{x}^n_{ir})}{\left(1 - \frac{x^n_{ir}}{c_r}\right)}\right\}. \tag{5.8}$$

For $n \geq N$, $i \in N$, and $r \in R_i$ we set

$$\bar{p}^n_r := \max\left\{p'_r, \max_{j\in N\setminus\{i\}} \frac{U'_j\left(\sum_{r\in R} \bar{x}^n_{jr}\right)c_r}{\left(1 - \frac{x^n_{jr}}{c_r}\right)}\right\}. \tag{5.9}$$

Define $\bar{p} = \lim_{n\to\infty} \bar{p}^n$. Then $\bar{p} = p'$, since for $r \in R \setminus R_f(x)$ we get

$$\bar{p}_r = \max\left\{p'_r, \max_{i\in N} \frac{U'_i\left(\sum_{r\in R} x'_{ir}\right)(c_r - x'_{ir})}{\left(1 - \frac{x_{ir}}{c_r}\right)}\right\} \text{ and } p'_r \geq \max_{i\in N} \frac{U'_i\left(\sum_{r\in R} x'_{ir}\right)(c_r - x'_{ir})}{\left(1 - \frac{x_{ir}}{c_r}\right)},$$

and for $r \in R_i$ we have

$$\bar{p}_r = \max\left\{p'_r, \max_{j\in N\setminus\{i\}} U'_j\left(\sum_{r\in R} x'_{jr}\right)c_r\right\} \text{ and } p'_r \geq \max_{j\in N\setminus\{i\}} U'_j\left(\sum_{r\in R} x'_{jr}\right)c_r.$$

It remains to argue that there exists $N' \geq N$ with $(\bar{x}^n, \bar{p}^n) \in \gamma(x^n)$ for all $n \geq N'$. It is clear that (5.2)-(5.4) are fulfilled. Moreover, (5.6) is fulfilled by construction of $\bar{p}^n$. For (5.5), note that if for $r \in R$ we have that $p'_r < \max\{2, 3/c_r\}c_r^2 \max_{i\in N} U'_i(0)$ or

$$p'_r > \begin{cases} \max_{i\in N} \frac{U'_i\left(\sum_{r\in R} x'_{ir}\right)(c_r - x'_{ir})}{\left(1 - \frac{x_{ir}}{c_r}\right)} & \text{if } r \in R \setminus R^f(x), \\ \max_{j\in N\setminus\{i\}} U'_j\left(\sum_{r\in R} x'_{jr}\right)c_r & \text{if } r \in R_i(x), \end{cases}$$

there exists $N' \geq N$ such that $\bar{p}^n_r \leq \max\{2, 3/c_r\} \cdot c_r^2 \max_{i\in N} U'_i(0)$ for all $n \geq N'$. Furthermore note that for $i \in N$ and $r \in R_i(x)$, we have

$$\max_{j\in N\setminus\{i\}} U'_j\left(\sum_{r\in R} x'_{jr}\right)c_r \leq \max_{i\in N} U'_i(0)c_r < \max\{2, 3/c_r\}c_r^2 \max_{i\in N} U'_i(0),$$

thus one of the above stated conditions is always fulfilled. Altogether, if there is no $N' \geq N$ such that $\bar{p}_r^n \leq \max\{2, 3/c_r\} \cdot c_r^2 \max_{i\in N} U_i'(0)$ is fulfilled for all $n \geq N'$, then r needs to be in $R \setminus R^f(x)$ and

$$\max\{2, 3/c_r\} c_r^2 \max_{i\in N} U_i'(0) = p_r' = \max_{i\in N} \frac{U_i'\left(\sum_{r\in R} x_{ir}'\right)(c_r - x_{ir}')}{\left(1 - \frac{x_{ir}}{c_r}\right)}. \tag{5.10}$$

Define for all $i \in N$ and $r \in R \setminus R^f(x)$ the values

$$v_{ir} := \frac{U_i'\left(\sum_{r\in R} x_{ir}'\right)(c_r - x_{ir}')}{\left(1 - \frac{x_{ir}}{c_r}\right)} \text{ and } b_r := \max\{2, 3/c_r\} c_r^2 \max_{i\in N} U_i'(0).$$

Note that if (5.10) is fulfilled for a resource $r \in R \setminus R^f(x)$, that is, $\max_{i\in N} v_{ir} = b_r$, then there is a unique player i with $v_{ir} = b_r$, since

$$\max\{2, 3/c_r\} c_r^2 \max_{i\in N} U_i'(0) = \frac{U_i'\left(\sum_{r\in R} x_{ir}'\right)(c_r - x_{ir}')}{\left(1 - \frac{x_{ir}}{c_r}\right)} \leq \frac{U_i'(0) c_r^2}{c_r - x_{ir}} \leq \frac{\max_{i\in N} U_i'(0) c_r^2}{c_r - x_{ir}}$$

and thus

$$x_{ir} \geq c_r - \frac{1}{\max\{2, 3/c_r\}} \geq \frac{2}{3} c_r > \frac{c_r}{2}.$$

Clearly, $x_{ir} > c_r/2$ can only hold for one player. Call this player the *critical* player for resource r. Now let R^c be the *critical* subset of resources, that is, the resources where there is no $N' \geq N$ such that $\bar{p}_r^n \leq \max\{2, 3/c_r\} \cdot c_r^2 \max_{i\in N} U_i'(0)$ for all $n \geq N'$. Since for each $r \in R^c$ there is a unique critical player, we may subdivide R^c into disjoint sets R_i^c for $i \in N$, where R_i^c denotes the critical resources for which player i is the critical player. We will now argue that by slightly changing the sequence $\bar{x}^n$ of allocations, we can ensure that $R^c = \emptyset$. To this end, take $i \in N$ and $r \in R_i^c$. We show how to adapt the sequence such that r is not critical anymore, and there are no new critical resources. This shows the claim.

Consider first the case that there exists a player $j \neq i$ with $x_{jr}' > 0$ and $v_{jr'} < b_{r'}$ for all $r' \in R \setminus R^f(x)$. Then, we may adapt the sequence as follows: Define $\varepsilon_n \geq 0$ as the smallest value such that

$$\frac{U_i'\left(\sum_{r\in R} \bar{x}_{ir}^n + \varepsilon_n\right)(c_r - \bar{x}_{ir}^n - \varepsilon_n)}{\left(1 - \frac{\bar{x}_{ir}^n}{c_r}\right)} \leq b_r.$$

Since $\frac{U_i'\left(\sum_{r\in R} \bar{x}_{ir}^n\right)(c_r - \bar{x}_{ir}^n)}{\left(1 - \frac{\bar{x}_{ir}^n}{c_r}\right)}$ converges to $v_{ir} = b_r$ and $\bar{x}_{jr}^n$ converges to $x_{jr}' > 0$, we know that ε_n converges to zero and there exists $N' \geq N$ with $\varepsilon_n \leq \bar{x}_{jr}^n$ for all $n \geq N'$. Redefine

$$\bar{x}_{ir}^n := \bar{x}_{ir}^n + \varepsilon_n,\ \bar{x}_{jr}^n := \bar{x}_{jr}^n - \varepsilon_n,$$

and adapt $\bar{p}^n$ according to (5.8) and (5.9) for all $n \geq N'$. Clearly, $(\bar{x}^n, \bar{p}^n) \to (x', p')$ and the conditions (5.2)-(5.4) and (5.6) are fulfilled for all $n \geq N'$. Moreover, since $v_{jr'} < b_{r'}$ for all $r' \in R \setminus R^f(x)$ and $U_j'(\sum_{r\in R} x_{jr}') c_{r'} < \max\{2, 3/c_{r'}\} \cdot c_{r'}^2 \max_{i\in N} U_i'(0)$ whenever $r' \in R^f(x) \setminus R_j(x)$, there exists $N'' \geq N'$ such that (5.5) is fulfilled for all resources in $(R \setminus R^c) \cup \{r\}$ and all $n \geq N''$. That is, resource r is not critical anymore, and there are no new critical resources.

In the second case, there exists a player $j \neq i$ with $x_{jr}' > 0$ and a resource $\tilde{r} \in R \setminus R^f(x)$ with $x_{i\tilde{r}}' > 0$ and $v_{i\tilde{r}} < b_{\tilde{r}}$. Define $\varepsilon_n \geq 0$ as the smallest value such that

$$\frac{U_i'\left(\sum_{r\in R} \bar{x}_{ir}^n\right)(c_r - \bar{x}_{ir}^n - \varepsilon_n)}{\left(1 - \frac{\bar{x}_{ir}^n}{c_r}\right)} \leq b_r.$$

Since $\frac{U_i'\left(\sum_{r\in R} \bar{x}_{ir}^n\right)(c_r - \bar{x}_{ir}^n)}{\left(1 - \frac{\bar{x}_{ir}^n}{c_r}\right)}$ converges to b_r, and $\bar{x}_{jr}^n$ converges to $x_{jr}' > 0$, and $\bar{x}_{i\tilde{r}}^n$ converges to $x_{i\tilde{r}}' > 0$, we know that ε_n converges to zero and there exists $N' \geq N$ with $\varepsilon_n \leq \min\{\bar{x}_{jr}^n, \bar{x}_{i\tilde{r}}^n\}$ for all $n \geq N'$. Redefine

$$\bar{x}_{ir}^n := \bar{x}_{ir}^n + \varepsilon_n, \bar{x}_{jr}^n := \bar{x}_{jr}^n - \varepsilon_n, \bar{x}_{i\tilde{r}}^n := \bar{x}_{i\tilde{r}}^n - \varepsilon_n, \bar{x}_{j\tilde{r}}^n := \bar{x}_{j\tilde{r}}^n + \varepsilon_n,$$

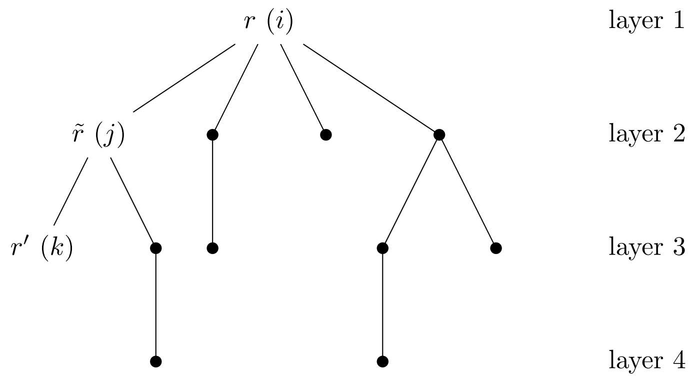

Figure 4: Illustration for the proof of Proposition 5.2. Some nodes are labelled by their name, followed by the unique critical player in brackets.

and adapt $\bar{p}^n$ according to (5.8) and (5.9) for all $n \geq N'$. With that, $(\bar{x}^n, \bar{p}^n) \to (x', p')$ and the conditions (5.2)-(5.4) and (5.6) are fulfilled for all $n \geq N'$. Moreover, since $v_{i\tilde{r}} < b_{\tilde{r}}$ and $v_{jr} < b_r$, there exists $N'' \geq N'$ such that (5.5) is satisfied for all resources in $(R \setminus R^c) \cup \{r\}$ and all $n \geq N''$.

It remains to assume that we have none of the above two cases. Note that since $v_{ir} = b_r > 0$, we have that $x'_{ir} < c_r$. Thus there exists a player $j \in N \setminus \{i\}$ with $x'_{jr} > 0$. We now adapt $\bar{x}^n$ and $\bar{p}^n$ as described in the first case, making r non-critical, but possibly creating new critical resources (by the lower sum $\sum_{r \in R} \bar{x}^n_{jr}$ for player j). If there are no new critical resources, we may stop the procedure. Otherwise, we analyze and treat each of the new critical resources in the same manner as we did with r. That is, consider any new critical resource $\tilde{r} \in R^c_j$. If there exists a player $k \neq j$ with $x'_{k\tilde{r}} > 0$ and $v_{kr} < b_r$ for all $r \in R \setminus R^f(x)$, we adapt the sequence as described in the first case (with $\tilde{r}, j, k$ in the place of r, i, j), and if there exists a player $k \neq j$ with $x'_{k\tilde{r}} > 0$ and a resource $r' \in R \setminus R^f(x)$ with $x'_{jr'} > 0$ and $v_{jr'} < b_{r'}$, we adapt the sequence as described in the second case (with $r', \tilde{r}, j, k$ in the place of $\tilde{r}, r, i, j$). In both cases, we get that $\tilde{r}$ is not critical anymore (while creating no new critical resources). If none of these two cases is applicable: Since $v_{j\tilde{r}} = b_{\tilde{r}} > 0$ and thus $x'_{j\tilde{r}} < c_{\tilde{r}}$, there exists a player $k \neq j$ with $x'_{k\tilde{r}} > 0$. Furthermore $k \neq i$ due to the fact that the second case above was not applicable during the treatment of resource r. We adapt $\bar{x}^n$ and $\bar{p}^n$ as described in the first case (with $\tilde{r}, j, k$ in the place of r, i, j), making $\tilde{r}$ non-critical, but possibly creating new critical resources (by the lower sum $\sum_{r \in R} \bar{x}^n_{kr}$ for player k). We treat each resource in R^c_j which was made critical due to the change on r in this manner. As a result, those resources are not critical anymore (however there might again be new critical resources). If there are no new critical resources we are done, otherwise we continue with the new critical resources in the described manner. It remains to show that this procedure has to stop, that is, at some point there are no new critical resources created. The crucial argument is that the number of players is finite. For a formal proof, let us arrange r and the resources which become critical in the procedure in a layered tree structure (see Figure 4 for illustration). The root of the tree is r, and we denote this as layer 1. For $\ell \geq 2$, layer ℓ contains all resources which are made critical by the treatment of a resource from layer $\ell - 1$, and a resource in layer ℓ is connected by an edge to the resource in layer $\ell - 1$ whose treatment made it critical. As an example, the second layer contains all resources which become critical due to the treatment of resource r, and are connected to r by an edge. We will now argue that there can be at most n layers, showing the claim. To this end, consider any resource r' contained in layer ℓ. There is a unique path P in the tree connecting r with r', and this path contains ℓ nodes. As each node corresponds to a critical resource, we may associate each node $\bar{r}$ with the unique player i having $v_{i\bar{r}} = b_{\bar{r}}$. We now prove that any player can be contained at most once in the sequence of players corresponding to path P, showing $\ell \leq n$. Assume, by contradiction, that there is a player j contained at least twice, with corresponding resources r'' and $\bar{r}$ where r'' appears in the smaller layer. Note that the treatment of resource r'' created new critical resources, showing that there is no resource $\tilde{r}$ with $x'_{j\tilde{r}} > 0$ and $v_{j\tilde{r}} < b_{\tilde{r}}$ (second case could not be applied when treating r''). However, the reason why $\bar{r}$ was added to the tree is that it was made critical by the treatment of its parent node $\tilde{r}$, and that implies $x'_{j\tilde{r}} > 0$ and $v_{j\tilde{r}} < b_{\tilde{r}}$; contradiction. This completes the proof of Claim 5.3. □

Altogether, we have thus shown Proposition 5.2. □

Using the above results, we can now show that accumulation points of $(x^k)_{k\geq 0}$ are fixed points of f. Note that this property can also be derived using a result from the theory of dynamical systems, namely the invariance principle of La Salle [19], but to apply this theorem we still need the Propositions 5.1 and 5.2.

PROPOSITION 5.3. *All accumulation points of the sequence $(x^k)_{k\geq 0}$ are fixed points of f.*

Proof. Let $\bar{x}$ be an accumulation point of $(x^k)_{k\geq 0}$ and $(x^{k_j})_{j\geq 0}$ a subsequence converging to $\bar{x}$. By the continuity of f (Proposition 5.2), the subsequence $x^{k_j+1} = f(x^{k_j})$ converges to $f(\bar{x})$. Since by Proposition 5.1 the total utility $U(x^k)$ converges, say to a value $\bar{U}$, and U is a continuous function, we have that $U(\bar{x}) = U(f(\bar{x})) = \bar{U}$. Again using Proposition 5.1, one finally gets that $f(\bar{x}) = \bar{x}$. ◻

We can now prove that the total utility $U(x^k)$ converges to the best-possible value $U^*(x^0)$. We start with the case that $x^0_{ir} < c_r$ for all $i \in N$ and $r \in R$, that is, $R'(x^0) = \emptyset$ and $U^*(x^0) = U^*$.

THEOREM 5.1. *Assume that $x^0_{ir} < c_r$ for all $i \in N$ and $r \in R$. Then, any accumulation point $\bar{x}$ of the sequence $(x^k)_{k\geq 0}$ is an optimal allocation (and, consequently, $U(x^k) \to U(\bar{x}) = U^*$).*

Proof. Define $\mu := \max_{i\in N} U_i'(\sum_{r'\in R} \bar{x}_{ir'})$. We show that $\bar{x}_{ir} > 0$ implies $U_i'(\sum_{r'\in R} \bar{x}_{ir'}) = \mu$. We know from Proposition 5.3 that $\bar{x}$ is a fixed point of f. By Proposition 5.1, this implies that $\bar{x}$ is socially optimal among all allocations x' with $x'_r = \bar{x}_r$ for all $r \in R' := \{r \in R : \bar{x}_{ir} = c_r \text{ for some } i \in N\}$. That is, $\bar{x}$ is an optimal solution for the following optimization problem:

$$
\begin{aligned}
\max \quad & \sum_{i\in N} U_i\left(\sum_{r\in R} x_{ir}\right) & \\
\text{s.t.:} \quad & \sum_{i\in N} x_{ir} \leq c_r & \forall r \in R,\\
& x_{ir} = \bar{x}_{ir} & \forall r \in R', i \in N,\\
& x_{ir} \geq 0, & \forall\, i \in N, r \in R.
\end{aligned}
$$

That is, for $r \in R \setminus R'$ we have that $\bar{x}_{ir} > 0$ implies $U_i'(\sum_{r'\in R} \bar{x}_{ir'}) = \mu$.

Now consider $r \in R'$ with $\bar{x}_{ir} = c_r$ and $j \neq i$. We need to show that $U_j'\left(\sum_{r'\in R} \bar{x}_{jr'}\right) \leq U_i'\left(\sum_{r'\in R} \bar{x}_{ir'}\right)$. Consider a subsequence $(x^{k_t})_{t\geq 0}$ converging to $\bar{x}$. For any $0 < \varepsilon < c_r/2$, there exists $\bar{t} = \bar{t}(\varepsilon)$ large enough so that for all $t \geq \bar{t}$ we have $x^{k_t}_{ir} \geq c_r - \varepsilon$ (and $x^{k_t}_{jr} \leq \varepsilon$). Note that $x^{k_t} \neq \bar{x}$ for any t, since $x^{k_t}_{ir} < c_r = \bar{x}_{ir}$ for all t. Thus, since $x^{k_t} \to \bar{x}$, we can choose $t' \geq \bar{t}$ with $x^{k_{t'+1}}_{ir} > x^{k_{t'}}_{ir}$. Now consider the set

$$\{x^{k_{t'}}_{ir}, x^{k_{t'}+1}_{ir}, \ldots, x^{k_{t'+1}}_{ir}\} = \{x^{k_{t'}+s}_{ir} : s \in T\} \text{ with } T := \{0, 1, \ldots, k_{t'+1} - k_{t'}\}.$$

Choose $s \in \operatorname{argmax}\{x^{k_{t'}+s}_{ir} : s \in T\}$. Note that $s \geq 1$ since $x^{k_{t'}}_{ir} < x^{k_{t'+1}}_{ir}$, and thus $x^{k_{t'}+s}_{ir} \geq x^{k_{t'}+s-1}_{ir}$. Furthermore, $x^{k_{t'}+s}_{ir} \geq x^{k_{t'+1}}_{ir} \geq c_r - \varepsilon$. To simplify notation a bit, let for the moment be $k := k_{t'} + s - 1$. We then have that $x^{k+1}_{ir} \geq c_r - \varepsilon > c_r/2 > 0$ and $x^{k+1}_{ir} \geq x^k_{ir}$, thus by the Nash conditions

$$U_i'\left(\sum_{r'\in R} x^{k+1}_{ir'}\right) = \frac{c_r - x^k_{ir}}{c_r - x^{k+1}_{ir}} \cdot \frac{p^{k+1}_r}{c_r} \geq \frac{p^{k+1}_r}{c_r}. \tag{5.11}$$

On the other hand, the Nash conditions also imply that

$$U_j'\left(\sum_{r'\in R} x^{k+1}_{jr'}\right) \leq \frac{c_r - x^k_{jr}}{c_r - x^{k+1}_{jr}} \cdot \frac{p^{k+1}_r}{c_r}.$$

Combining this with (5.11), and using that $x^k_{jr} \geq 0$ and $x^{k+1}_{jr} \leq \varepsilon$, we obtain that

$$U_j'\left(\sum_{r'\in R} x^{k+1}_{jr'}\right) < \frac{c_r - x^k_{jr}}{c_r - x^{k+1}_{jr}} \cdot U_i'\left(\sum_{r'\in R} x^{k+1}_{ir'}\right) \leq \frac{c_r}{c_r - \varepsilon} \cdot U_i'\left(\sum_{r'\in R} x^{k+1}_{ir'}\right).$$

We now redefine $x^{k_t} := x^{k_t+s}$ for $t \geq t'$. Note that the new subsequence still converges to $\bar{x}$ (we know that $x^{k_t+1} \to \bar{x}$). With this we have shown

$$U_j'\left(\sum_{r'\in R} x_{jr'}^{k_{t'}}\right) < \frac{c_r}{c_r - \varepsilon} \cdot U_i'\left(\sum_{r'\in R} x_{ir'}^{k_{t'}}\right).$$

By taking $\varepsilon \to 0$, the continuity of U_i' and U_j' finally imply that

$$U_j'\left(\sum_{r'\in R} \bar{x}_{jr'}\right) \leq U_i'\left(\sum_{r'\in R} \bar{x}_{ir'}\right),$$

as desired. ∎

Now turn to the case that $x_{ir}^0 = c_r$ for some $i \in N$ and $r \in R$, that is, $R'(x^0) = \{r \in R : x_{ir}^0 = c_r \text{ for some } i \in N\} \neq \emptyset$. Note that in this case, we may just consider the game with resource set $\bar{R} := R \setminus R'(x^0)$ and utilities $\bar{U}_i(t) := U_i(t + \sum_{r\in R_i} c_r)$, where $R_i := \{r \in R'(x^0) : x_{ir}^0 = c_r\}$ for all $i \in N$, and apply Theorem 5.1 to achieve the following corollary.

Corollary 5.1. *For $x^0 \in S$, denote by $U^*(x^0)$ the best-possible total utility that can be achieved if the allocations for resources in $R'(x^0)$ are fixed as in x^0. Then, $U(x^k) \to U^*(x^0)$.*

We have thus shown that the total utility $U(x^k)$ converges to the best-possible value (w.r.t. the initial allocation x^0). However note that in general, the system optimal solution is not unique in terms of resource-specific allocations (since we only care about the *aggregated* resource shares of a player), and thus the above result does not immediately imply convergence of the resource-specific allocations x^k. However if the best-possible *aggregated* allocation is unique (that is, there is a unique vector $\ell^* \in \mathbb{R}_{\geq 0}^N$ with $\sum_{i\in N} \ell_i^* = \sum_{r\in R} c_r$ and $U(\ell^*) := \sum_{i\in N} U_i(\ell_i^*) = U^*(x^0)$), we easily get convergence of the aggregated allocation

$$\ell(x^k) := \left(\sum_{r\in R} x_{ir}^k\right)_{i\in N}$$

to ℓ^*, see the following theorem.

Theorem 5.2. *Let $x^0 \in S$. If there is a unique best-possible* aggregated *allocation ℓ^* (for instance, if all utility functions are strictly concave), the aggregated allocation $\ell(x^k)$ converges to ℓ^*.*

Proof. Let ℓ^* be the unique best-possible aggregated allocation vector. Let $\bar{\ell}$ be an accumulation point of the sequence $(\ell(x^k))_{k\geq 0}$. Then, there exists a convergent subsequence $(x^{k_j})_{j\geq 0}$ with $x^{k_j} \to \bar{x}$ and $\ell(\bar{x}) = \bar{\ell}$. We then get from Lemma 5.1 that $U(\bar{x}) = U(x^0)$ is best-possible, which shows that $\bar{\ell} = \ell(\bar{x}) = \ell^*$. ∎

Remark 5.1. *Note that if we only have a single good, then $\ell(x) = x$. Consequently, for a single good and strictly concave utilities we get convergence of the allocation x^k to a best-possible solution.*

In the subsequent theorem, we treat the case of linear utilities, and show that the sequence of allocations $(x^k)_{k\geq 0}$ converges to a best-possible allocation (again with respect to the initial endowment x^0). Note that for linear utilities, the analysis simplifies since different resources are not coupled anymore and one can treat each resource separately. As stated above in Remark 5.1, for a single good the aggregated allocation is just the allocation itself, which allows us to prove convergence of x^k, instead of only $\ell(x^k)$. However, we cannot just use Theorem 5.2, since uniqueness of the best-possible allocation is not guaranteed anymore (there might be more than one player having highest slope).

Theorem 5.3. (Convergence for linear utilities) *Assume that the utility functions $U_i, i \in N$, are linear, that is, $U_i(z) = a_i z_i$ with $a_i > 0$ for all $i \in N$. Then, the sequence $(x^k)_{k\geq 0}$ converges (to a best-possible allocation w.r.t. x^0).*

Proof. As explained above, it suffices to show the theorem for a single resource, $R = \{r\}$ (the general case follows since resources are not coupled for linear utilities and can thus be handled separately). Furthermore, if the sequence $(x^k)_{k\geq 0}$ converges, we immediately get from Corollary 5.1 that the limit is best-possible. Finally, it suffices to show convergence for the case that $x_{ir}^0 < c_r$ for all $i \in N$ (otherwise convergence is clear since the allocation of (the single) resource r remains fixed throughout the sequence). In order to show convergence, consider an (arbitrary) accumulation point $\bar{x}$. We show that $\bar{x}$ is uniquely defined, thus there is only one accumulation point and the sequence converges: We know from Theorem 5.1 that $\bar{x}$ is optimal. With $a^* := \max_{i\in N} a_i$ and $N^* := \{i \in N : a_i = a^*\}$, we have that $\bar{x}_{ir} = 0$ for all $i \notin N^*$. It thus remains to consider the players in N^*. But as we show below, the fraction x_{ir}^k of good r that player $i \in N^*$ gets is monotonically increasing in k, thus x_{ir}^k converges to some value $\hat{x}_{ir}$, and consequently $\bar{x}_{ir} = \hat{x}_{ir}$. It remains to show that x_{ir}^k is monotonically increasing for $i \in N^*$. Consider Claim 5.1. By the second statement, we get for any $i \in N$ that if $x_{ir}^{k+1} > x_{ir}^k$, then $p_r^{k+1} < a_i c_r$. Since $x^{k+1} \neq x^k$ (else we immediately get convergence), there exists a player $i \in N$ who gets more, and thus $p_r^{k+1} < a_i c_r \leq a^* c_r$. But by contraposition of the first inequality of Claim 5.1, namely that $x_{ir}^{k+1} \geq x_{ir}^k$ if $p_r^{k+1} \leq a_i c_r$, this shows $x_{ir}^{k+1} \geq x_{ir}^k$ for $i \in N^*$, and completes the proof. ∎

To conclude this section, we summarize our results with respect to the equilibrium dynamics. The total utility always converges to the best-possible value w.r.t. the initial endowment. If the best-possible aggregated allocation is unique (which for instance holds if all utilities are strictly concave), we furthermore get convergence of the sequence of aggregated allocations. Finally, if all utility functions are linear, we even get convergence of the resource-specific allocations.

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