Proportional Apportionment: A Case Study From the Chilean Constitutional Convention

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ABSTRACT

How to elect the representatives in legislative bodies is a question that every modern democracy has to answer. This design task has to consider various elements so as to fulfill the citizens' expectations and contribute to maintaining a healthy democracy. In this work, we present five proposals of electoral methods based on proportional representation that extend the notion of proportionality to several dimensions: For instance, simultaneous proportionality in political, geographical, and gender representation. We also consider including two additional desirable properties for electoral methods: A minimum threshold to obtain political representation, and the incorporation of plurality voting, guaranteeing the election of the highest voted candidate in each district. We use the Chilean Constitutional Convention election (May 15-16, 2021) results as a testing ground, and compare the apportionment obtained under each method according to four criteria: Proportionality, representativeness, robustness, and voting power. We conclude that it is feasible to design and implement an electoral method satisfying all mentioned properties. Our findings may be useful in assessing electoral designs in other contexts as well.

CCS CONCEPTS

• **Mathematics of computing → Combinatorial optimization**; • **Theory of computation → Rounding techniques**; **Integer programming**.

KEYWORDS

Apportionment, Electoral Design, Social Choice

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1 INTRODUCTION

On October 18, 2019, a civil unrest broke out in Chile. By far the largest since Chile's return to democracy in 1989, the movement was motivated by the high inequality levels in modern Chile. Interestingly, the economic inequality, measured for instance using the Gini index, has been slowly, but steadily, going down in the last 30 years. However, the perceived disparity finds its roots deeper in social norms, opportunities, and essential needs such as education, health, and pensions. The protests brought the country to a critical point, and an attempt to funnel the unrest and provide a democratic way out was seen as urgent. Thus, a month later, on November 15, the country's leading political forces reached an agreement to write a new political constitution.

The agreement was reaffirmed by a referendum approved with 78% of the votes on October 25, 2020. On May 15-16, 2021, there was a voting process to elect the members of the Constitutional Convention, the body in charge of writing the new political constitution of the country. Furthermore, this election drew attention by including two constraints on the representative body to be elected. First, 17 of the 155 seats of the convention were reserved for ten different ethnic groups.[1] And second, a remarkable feature of the election was to incorporate gender balance in the convention, as a way to ensure women's participation in the convention.[2]

The Chilean case is not an exception. As modern societies become more complex, incorporating dimensions beyond the classical political and geographical aspects has turned into a growing need. For instance, besides the Chilean case, New Zealand's parliament has included ethnic representation for more than 50 years, while Bosnia and Herzegovina's Parliament was proposed to include a division of three types of "Constituent People": Bosniacs, Croats and Others [9].

In this paper, we explore and develop methods to handle these multiple objectives when electing a house of representatives. Our working ground is the 2021 Chilean Constitutional Convention election, and the methods we explore are strongly based on the idea of proportionality. Proportional representation lies at the heart of fair representation, and its origins are from the 18th century with the development of the Jefferson/D'Hondt method and the Webster/Sainte-Laguë method. The main idea of these methods is to scale the votes obtained by each party in an election by a common value, called *multiplier*, and then the results are rounded to meet

[1] In Chile around two million people identify themselves as part of these ethnic groups, according to the 2017 census.

[2] Women have been historically underrepresented in Chilean politics, for instance, today they hold 23% of the seats in the parliament.

the total number of seats to be allocated. There is a long and rich body of literature for the apportionment problem and the divisor methods, intersecting different areas such as operations research, computer science, and political science. For a formal treatment of the theory and a historical survey, we refer to the book by Balinski and Young [7] and to the recent book by Pukelsheim [13]. For a deeper treatment of social choice and new methods, we also refer to the book and article by Balinski and Laraki [5, 6] and to the recent book by Serafini [14].

In their seminal work, Balinski and Demange extend the notion of proportionality and divisor methods to the case in which the apportionment is ruled by two dimensions, e.g., political parties and geographic districts, studying this extension from an axiomatic and algorithmic point of view [2, 3]. Balinski [1, 4] also proposed variants of this method, while Maier et al. [12] conducted a real-life benchmark study of biproportional apportionment and its variants. Recently, Cembrano et al. [8] extended this approach to the case of an arbitrary number of dimensions, showing that, although proportional apportionments do not always exist, they do if small deviations are allowed.

Our contribution. In this paper, we propose and study the properties of several electoral methods with gender balance based on bi- and three-dimensional proportionality, and we use the Chilean Constitutional Convention election as a testing ground and basis for the comparison. We introduce additional features as a threshold over the votes obtained by a list in order to be eligible, and the election of the top candidate of each district. In particular, we extend the existential and algorithmic result by Cembrano et al. to incorporate the plurality constraints as part of the method. As relevant findings, we observe that deviations from the prescribed marginals in three-proportional methods are significantly smaller than those predicted by the theoretical guarantee. In addition, three-proportional methods induce a fundamental trade-off between local and global proportionality, and they are more robust and closer to the well-known principle of *one person, one vote.* Overall, the election of top-voted candidates combined with a threshold over the votes obtained by a list appears as a reasonable midpoint while ensuring high levels of representativeness.

2 PRELIMINARIES

In the three-dimensional apportionment problem, there is a set of districts D, a set of lists L, and a set of genders G. There is also a set of candidates C, and each candidate $c \in C$ competes in a district $\mathbf{d}(c) \in D$, belongs to a list $\boldsymbol{\ell}(c) \in L$ and is of gender $\mathbf{g}(c) \in G$. The results of the election consist in the votes $v(c)$ each candidate c obtains. In the problem, we are also given a vector q containing the number of seats to be assigned in each district $d \in D$, and a strictly positive integer number H corresponding to the total number of seats to be allocated, also called the *house size.* Since we are often interested in the aggregated votes when defining algorithms and evaluating their performance, we construct a matrix $\mathcal{V}$ such that $\mathcal{V}_{d\ell g}$ is the sum of the votes of the candidates of list ℓ, gender g, in district d, i.e.

$$\mathcal{V}_{d\ell g} = \sum_{\substack{c \in C:\, \mathbf{d}(c)=d \\ \boldsymbol{\ell}(c)=\ell,\, \mathbf{g}(c)=g}} v(c).$$

The objective of the methods discussed in this work is to find an assignment of candidates to seats represented by a function χ, so $\chi(c) = 1$ if candidate c is elected and $\chi(c) = 0$ otherwise.

2.1 The Jefferson/D'Hondt Method

For the allocation of seats across the lists, we describe in what follows the Jefferson/D'Hondt method of apportionment. Given an integer vector $\mathcal{P}$ containing the votes of each list $\ell \in L$ and a house size H, we say that x is an apportionment according to the Jefferson/D'Hondt method if $\sum_{\ell \in L} x_\ell = H$ and there exists $\lambda > 0$ such that $x_\ell \in [\![\lambda \mathcal{P}_\ell]\!]$ for each list $\ell \in L$, where $[\![\cdot]\!]$ denotes the downwards rounding operation.[3] It is possible to extend this notion of apportionment to the case when we are given lower and/or upper bounds in the number of seats a list should get. If I_ℓ and U_ℓ represent the minimum and maximum number of seats list ℓ should get, respectively, we say that a vector x is an apportionment according to the Jefferson/D'Hondt method with bounds if there exists $\lambda > 0$ such that

$$\sum_{\ell \in L} x_\ell = H \text{ and } x_\ell \in \operatorname{mid}\{I_\ell, [\![\lambda \mathcal{P}_\ell]\!], U_\ell\} \text{ for each } \ell \in L, \qquad (1)$$

where $\operatorname{mid}\{a, b, c\}$ represents the median value between a, b and c. Note that if $I_\ell = 0$, the last is equivalent to $x_\ell \in \min\{[\![\lambda \mathcal{P}_\ell]\!], U_\ell\}$, and if $U_\ell = H$, it is equivalent to $x_\ell \in \max\{I_\ell, [\![\lambda \mathcal{P}_\ell]\!]\}$. The described apportionments are guaranteed to exist and can be found through combinatorial algorithms or linear programming. For a deep treatment of the theory of apportionment, we refer to the book of Balinski and Young [7].

2.2 Biproportionality and Three-proportionality

It is possible to extend the one-dimensional approach to the cases of two and three dimensions, which are particularly relevant for some of the methods discussed in the following sections. In the bidimensional context, we are given an integer matrix $\mathcal{V}$ with entries in $L \times G$, containing the votes of the candidates of each list ℓ and gender g, a vector r containing the number of seats to be assigned to each list ℓ, a vector s containing the number of seats to be assigned to each gender g, and a house size H. We say that an integer matrix x is a biproportional apportionment if there exist strictly positive values μ_ℓ for each list ℓ and γ_g for each gender g such that the following holds:

$$\sum_{g \in G} x_{\ell g} = r_\ell \quad \text{for each list } \ell \in L, \qquad (2)$$

$$\sum_{\ell \in L} x_{\ell g} = s_g \quad \text{for each gender } g \in G, \qquad (3)$$

$$x_{\ell g} \in [\![\mu_\ell \gamma_g \mathcal{V}_{\ell g}]\!] \quad \text{for every } \ell \in L \text{ and } g \in G. \qquad (4)$$

Under mild conditions, such apportionment is guaranteed to exist, and once again, it can be found by solving a linear program. We refer to the works of Balinski and Demange for the technicalities about this method and algorithms for finding such biproportional apportionments [2, 3].

[3] That is, $[\![0]\!] = \{0\}$, $[\![t]\!] = \{n\}$ when $t \in (n, n+1)$ and $[\![t]\!] = \{n-1, n\}$ when $t = n > 0$.

Recently, Cembrano et al. studied the extension of the biproportional method to the case of arbitrary dimension [8]. More formally, in the three-dimensional case we are given an integer matrix $\mathcal{V}$ with entries in $D \times L \times G$, containing the votes of the candidates of each list ℓ and gender g in district d, a vector q containing the number of seats to be assigned to each district d, a vector r containing the number of seats to be assigned to each list ℓ, a vector s containing the number of seats to be assigned to each gender g, and a house size H. We say that an integer matrix x is a u-approximate three-proportional apportionment if there exist strictly positive values λ_d for each district d, μ_ℓ for each list ℓ and γ_g for each gender g such that the following holds:

$$q_d - u_D \leq \sum_{\ell \in L} \sum_{g \in G} x_{d\ell g} \leq q_d + u_D \quad \text{for each district } d \in D, \tag{5}$$

$$r_\ell - u_L \leq \sum_{d \in D} \sum_{g \in G} x_{d\ell g} \leq r_\ell + u_L \quad \text{for each list } \ell \in L, \tag{6}$$

$$s_g - u_G \leq \sum_{d \in D} \sum_{\ell \in L} x_{d\ell g} \leq s_g + u_G \quad \text{for each gender } g \in G, \tag{7}$$

$$x_{d\ell g} \in [\![\lambda_d \mu_\ell \gamma_g \mathcal{V}_{d\ell g}]\!] \text{ for every } d \in D, \ell \in L \text{ and } g \in G. \tag{8}$$

If a matrix x verifies this definition with $u_D = u_L = u_G = 0$, we just say that x is a three-proportional apportionment for this instance. Cembrano et al. [8, Theorem 4] proved that, under mild conditions[4], a u-approximate three-proportional apportionment is guaranteed to exist for a given instance whenever $1/(u_D + 2) + 1/(u_L + 2) + 1/(u_G + 2) \leq 1$ and it can be found by using linear programming techniques. For instance, the above condition is satisfied when $u_D = u_L = u_G = 1$, or when $u_D = 0$, $u_L = 1$ and $u_G = 4$.

3 DESCRIPTION OF THE ELECTORAL APPORTIONMENT METHODS

3.1 The Chilean Constitutional Convention Method (CCM)

In what follows we describe the method used in the recent Chilean Constitutional Convention election on May 15-16, 2021. In a first step, the seats of each district $d \in D$ are divided between the lists and independent candidates according to the single-dimensional Jefferson/D'Hondt method, using the votes obtained by all the candidates of each list in the district. We obtain in this way a value r_ℓ for each $\ell \in L$. Then, the r_ℓ seats assigned to each list $\ell \in L$ are divided between its sublists, usually political parties, through the same method and provisionally assigned to the candidates of these sublists with more individual votes. If at this point the set of elected candidates achieve gender balance—meaning the same number of men and women if the number of seats of the district is even and at most one more man/woman if it is odd—the seats are assigned to these candidates.

Otherwise, the following procedure is repeated until the gender balance condition is satisfied: Pick the provisionally elected candidate of the over-represented gender with the lowest number of votes, and assign in his/her place the provisionally non-elected candidate of the other gender and his/her same party—or list if the former is not possible—with the highest number of individual votes. We refer to the Chilean electoral laws [10, 11] for the legal description of the method.

[4]This conditions are given by the feasibility of an LP, see Appendix A and the full version of this paper for details.

3.2 The Biproportional Method per District (BPM)

In this method, we run the following procedure for every district $d \in D$. In a first step, we find a 1-dimensional apportionment of seats to the lists in L by using the Jefferson/D'Hondt method, according to their votes, and we call r_ℓ the number of seats obtained by each list $\ell \in L$. Recall that the district marginals are given by q_d for each $d \in D$.

For each district $d \in D$ we set the gender marginals s as follows. For each gender $g \in G$ consider the total number of votes obtained by the candidates of gender g in the district, that is, $\sum_{\ell \in L} \mathcal{V}_{d\ell g}$. Then, we set $s_g = \lceil q_d/2 \rceil$ to the gender g garnering the majority of votes, and we set $\lfloor q_d/2 \rfloor$ to the other gender. Observe that when q_d is even we have that both genders obtain the same number of seats $q_d/2$, and when q_d is odd the seats unbalance is equal to one. Then, we compute a biproportional apportionment x satisfying (2)-(4) to obtain values $x_{\ell g}$ for each list $\ell \in L$ and each gender $g \in G$. The value $x_{\ell g}$ represents the total number of candidates elected from list ℓ and gender g in district d. Finally, the assignment of elected candidates is computed as follows: For every list $\ell \in L$ and for every gender $g \in G$ we set $\chi(c) = 1$ for the $x_{\ell g}$ top-voted candidates c such that $\mathbf{d}(c) = d$, $\boldsymbol{\ell}(c) = \ell$, and $\mathbf{g}(c) = g$. We set $\chi(c) = 0$ otherwise.

3.3 The Three-proportional Method (TPM)

In contrast to the previous methods, the three-proportional method produces a global apportionment instead of one apportionment per district. As described in the Section 2.2, a three-proportional apportionment does not exist in every case, but a u-approximate three-proportional apportionment is guaranteed to exist as long as $1/(u_D + 2) + 1/(u_L + 2) + 1/(u_G + 2) \leq 1$. In particular, this holds when $u_D = 1$, $u_L = 0$ and $u_G = 4$.

We now describe the method. The total number of seats is H and the district marginals are given by q_d for each $d \in D$. The list marginals are computed according to the Jefferson/D'Hondt method using for each list $\ell \in L$ the total number of votes $\sum_{d \in D} \sum_{g \in G} \mathcal{V}_{d\ell g}$, and let r_ℓ be the number of seats obtained by each list $\ell \in L$. The gender marginals are given as follows. For each gender $g \in G$ consider the total number of votes obtained by the candidates of gender g, that is, $\sum_{d \in D} \sum_{\ell \in L} \mathcal{V}_{d\ell g}$. Then, we set $s_g = \lceil H/2 \rceil$ to the gender g garnering the majority of votes, and we set $\lfloor H/2 \rfloor$ to the other gender. Observe that when H is even we have that both genders obtain the same number of seats $H/2$, and when H is odd the seats unbalance is equal to one.

(a) Initially, consider $u_L = 0$, $u_D = 0$ and $u_G = 0$. We check if there exists a three-proportional apportionment x satisfying (5)-(8) to obtain values $x_{d\ell g}$ for each district $d \in D$, each list $\ell \in L$ and each gender $g \in G$. If such apportionment exists, the assignment of elected candidates is computed as follows: For every district $d \in D$, for every list $\ell \in L$ and for every gender $g \in G$ we set $\chi(c) = 1$ for the top $x_{d\ell g}$ candidates c

such that $\mathbf{d}(c) = d$, $\boldsymbol{\ell}(c) = \ell$ and $\mathbf{g}(c) = g$. We set $\chi(c) = 0$ otherwise.

(b) Otherwise, we iteratively increment the value of u_G by one, we check in each case if there exists a solution satisfying (5)-(8), and we stop once an integral apportionment is found. The elected candidates are assigned in the same manner as before. Finally, if the described steps do not succeed with $u_G = 4$, we repeat the procedure by initially setting $u_L = 0$ and $u_D = 1$ and iteratively increasing u_G from zero to four. This procedure is guaranteed to find a solution.

3.4 The Three-proportional Method With Threshold (TPM3)

In this method, we include a threshold on the percentage of votes obtained by a list in order to be eligible for the seat apportionment. More specifically, we only include in the process the set of lists $\ell \in L$ that obtain at least a 3% of the votes, that is

$$\frac{\sum_{d\in D}\sum_{g\in G}\mathcal{V}_{d\ell g}}{\sum_{d\in D}\sum_{\ell'\in L}\sum_{g\in G}\mathcal{V}_{d\ell' g}} \geq 0.03. \tag{9}$$

In order to implement this constraint we compute the list marginals using the Jefferson/D'Hondt method with bounds by setting $U_\ell = 0$ for every list ℓ that do not satisfy condition (9), and $U_\ell = H$ for every list ℓ meeting the condition (9). For every list $\ell \in L$ we set $I_\ell = 0$. In particular, lists not meeting the condition (9) are given zero seats by the Jefferson/D'Hondt method. Then, we run the method described in Section 3.3.

3.5 The Three-proportional Method With Plurality Election (TPP)

In this method, we ensure that the top candidate of each district is elected by incorporating a set of constraints at the moment of computing a three-proportional apportionment. This feature was included by Maier et al. [12] in their real-life benchmark study of biproportional apportionments. Nevertheless, no theoretical study about the existence and computation of such apportionment was provided. We show that the results from Cembrano et al. can be extended when this plurality constraint is incorporated, see Appendix A and the full version of this paper for details.

We now describe the method. The district marginals are given by q_d for each district $d \in D$. For every list $\ell \in L$, let m_ℓ be the number of candidates from this list that are top candidates in their district. The list marginals r_ℓ are computed using the Jefferson/D'Hondt method with bounds by setting $U_\ell = H$ and $I_\ell = m_\ell$ for every list ℓ. The goal of electing the top-voted candidate of each district is imposed with the following constraints: For every top candidate c of a district, we have

$$x_{d\ell g} \geq 1 \text{ if } \mathbf{d}(c) = d, \boldsymbol{\ell}(c) = \ell \text{ and } \mathbf{g}(c) = g. \tag{10}$$

In order to incorporate these constraints into the notion of proportionality, we consider the value $\mathbb{1}_{d\ell g}$ that is equal to one if the top voted candidate of district d belongs to list ℓ and gender g, and zero otherwise. In this context we replace the proportionality condition (8) by the following condition:

$$\begin{gathered}\text{For every } d \in D, \ell \in L \text{ and } g \in G \text{ we have}\\ x_{d\ell g} \in [\![\lambda_d\mu_\ell\gamma_g\mathcal{V}_{d\ell g}]\!] \text{ if } \lambda_d\mu_\ell\gamma_g\mathcal{V}_{d\ell g} \geq \mathbb{1}_{d\ell g},\\ \text{and } x_{d\ell g} = 1 \text{ if } \lambda_d\mu_\ell\gamma_g\mathcal{V}_{d\ell g} < \mathbb{1}_{d\ell g}.\end{gathered} \tag{11}$$

Observe that when the top voted candidate of district d does not belong to list ℓ and gender g, condition (11) is equivalent to (8) for that tuple (d, ℓ, g). We say that a solution x satisfying (5)-(7) and (11) is a u-approximate three-proportional apportionment with plurality. In this method, we run the same steps (a)-(b) described in Section 3.3 considering u-approximate three-proportional apportionment with plurality instead.

3.6 The Three-proportional Method With Threshold and Plurality Election (TPP3)

In this method, we also ensure that the top candidate of each district is elected but we further consider the threshold condition (9). The district marginals are given by q_d for each district $d \in D$. For every list $\ell \in L$, let m_ℓ be the number of candidates from this list that are top candidates in their district. The list marginals are computed using the Jefferson/D'Hondt method (1) by setting $I_\ell = U_\ell = m_\ell$ for every list ℓ that do not satisfy condition (9), and we set $U_\ell = H$ and $I_\ell = m_\ell$ for every list ℓ meeting condition (9). Then the procedure runs in the same way as TPP in Section 3.5.

4 RESULTS AND ANALYSIS

In this section, we study the outcomes of the methods described above and evaluate their performance in terms of the proportionality of the results, the representativeness of the elected candidates, the robustness of the house configuration, and the value of each cast vote.

Chile's electoral map is divided into 28 electoral districts with a specified number of seats to be allocated in each district. In total 155 seats were to be allocated, 17 of which were reserved for ethnic minority groups, so that 138 seats were allocated to the 28 districts. Our comparison only considers these non-ethnic seats, because the ethnic seats were elected in a parallel election regardless of political and geographic distribution. We also mention that each voter votes for at most one candidate of his/her district.

In the recent Chilean Constitutional Convention election a total of 70 lists, including over 1300 candidates, competed for these 138 seats. Three of these lists correspond to well-established political alliances. The XP list represented the right-wing parties, including not only the traditional parties *Renovación Nacional* and *Unión Demócrata Independiente*, but also the newer centrist *Evopoli* and the extreme right *Partido Republicano*. The YB list represented the center-left parties that have mostly governed Chile in the last three decades, including the *Democracia Cristiana* and the *Partido Socialista*. The third list is the YQ list and corresponds to the left-wing parties such as the *Partido Comunista* and a number of much newer parties. Additionally, two important politically independent players in the election arose as conglomerates encompassing different lists—but did not compete in any district. These correspond to what we denote by LP (for *Lista del Pueblo*) and INN (for *Independientes No*

Neutrales).[5] [6] By observing the outcome of the methods previously described, we have that among the 70 lists and 28 independent candidates, only 20 lists and two independent candidates[7] obtain enough votes to be elected in either method. For ease of exposition, when presenting the results we omit the votes of the other lists and independent candidates, none of which obtained more than 0.51% of the votes and jointly represent less than the 10% of them. Note that the results are not affected by this modification. We also remark that the necessary deviations for obtaining apportionments in the three-proportional methods were of just one seat for TPM3 and TPP3, and no deviation was necessary in the case of TPM and TPP.

4.1 Proportionality

Proportionality in the election results is measured through the deviation of the political distribution from the perfectly fair distribution, a.k.a. *fair share* in the literature, which assigns to each list the (possibly fractional) number of seats that corresponds to the proportion of the house that the votes obtained represent. Figure 1 in the Appendix provides a graphical view of the political representation obtained by the lists under each method and this perfectly fair political distribution, and complete data is contained in the full version of this paper. It is seen that the global methods, especially those without a threshold (TPM and TPP), generate a political distribution much closer to the fair share than the local methods (CCM and BPM), with a smaller over-representation of the most voted list and an assignment of seats to the top-voted lists. The result is a highly varied parliament made up of large political blocks together with multiple lists that obtained a single representative. It is particularly relevant to remark that CCM does not assign any seat to list XA, which is the sixth most voted list with almost 4% of the votes, while the global methods allocate five seats to this list. The TPP3 method in particular balances the representation of the lists as a consequence of the election of pluralities, so that it does not allow the entry of multiple lists with a dispersion of votes at the national level, but it does allow strong local projects. In particular, the XM list, which is a project of the southernmost part of Chile (Magallanes), as well as the independents from districts 1 and 9, which were the first majorities of their districts, enter the parliament.

The notion of closeness or dispersion with respect to the fair share can be formalized through the Euclidean distance between an apportionment and the fair share, a measure commonly known as Gallagher Index in the political science and political economy literature. This notion is easily extended to the apportionment of a single district as well, comparing the political distribution of the seats assigned in the district with the fair distribution according to the votes.[8] Furthermore, we can define a three-dimensional Gallagher Index in order to evaluate the proportional allocation of seats across districts, lists, and genders simultaneously, simply as the Euclidean distance between the apportionment and the three-dimensional fair share. Following the idea of a three-proportional apportionment but now with the chance of fractional values in order to respect exact proportionality, a tensor f is a three-dimensional fair share if there exist values λ_d for each district d, μ_ℓ for each list ℓ and γ_g for each gender g such that $\sum_{\ell\in L}\sum_{g\in G} f_{d\ell g} = q_d$ for each district $d \in D$, $\sum_{d\in D}\sum_{g\in G} f_{d\ell g} = r_\ell$ for each list $\ell \in L$, $\sum_{d\in D}\sum_{\ell\in L} f_{d\ell g} = s_g$ for each gender $g \in G$, and $f_{d\ell g} = \lambda_d\mu_\ell\gamma_g\mathcal{V}_{d\ell g}$ for every district $d \in D$, list $\ell \in L$ and gender $g \in G$. Such tensor can be found by solving a convex optimization program, where the objective function which is minimized is the function

$$\sum_{d\in D}\sum_{\ell\in L}\sum_{g\in G} f_{d\ell g}\left(\log\left(\frac{f_{d\ell g}}{\mathcal{V}_{d\ell g}}\right) - 1\right).$$

By the KKT optimality conditions it can be verified that the optimal solution of such program is a three-dimensional fair share.

	CCM	**BPM**	**TPM**	**TPM3**	**TPP**	**TPP3**
Global GI	4.6	4.6	1.8	3.8	1.3	2.9
Local GI avg.	13.7	13.7	18.5	19.0	18.2	18.5
3-dim GI	3.9	3.8	3.6	3.8	3.6	3.8

Table 1: Gallagher Index (%) computed for the global political distribution, average of the local political distribution indexes, and three-dimensional index.

Table 1 shows the results obtained from the indices mentioned by method. Naturally, the district averages show that local methods (CCM and BPM) are closer to the local fair share than the others, which is essentially a property of the design since they achieve local proportionality. However, when summing up the results by district, the local errors generated start to add up and the distortion with respect to the fair share, summarized in the global Gallagher Index, increases. On the other hand, TPM and TPP are designed to achieve global proportionality so that the national results are much closer to the fair share. In fact, TPP ends up with the least deviation from political representation followed by the TPM, both being considerably better than the rest of the methods. In terms of the three-dimensional index, the results are similar between all the methods, with differences of at most 0.3 percentage points. This similarity can be explained due to the fact that the three-proportional methods generate a good allocation to lists but generate local distortions. Conversely, the national allocation to lists does not adjust correctly to the votes in CCM and BPM but the district allocation does. There is, therefore, a fundamental trade-off between local and global political representation.

4.2 Representativeness

In this section, we analyze the average of votes and percentages obtained by the elected candidates under each method, summarized in Table 2.

By construction, all methods that involve correction mechanisms in order to ensure gender parity imply a certain degree of loss of votes, due to the fact that candidates who may have been elected without corrections are substituted by other non-elected ones with a lower number of votes. A relevant observation is the fact that

[5]This association is standard as reported, for instance, by https://2021.decidechile.cl/#/ev/2021. Full election data can be found on the website of the Chilean Servicio Electoral (SERVEL) https://www.servel.cl/.

[6]When presenting the results for the remaining lists we use the election codes.

[7]These independent candidates are denoted as IND1 and IND9 because of the number of the districts where they participated.

[8]The Gallagher Index computed by district can be found in the full version of this paper.

	CCM	BPM	TPM	TPM3	TPP	TPP3
Avg. votes	14126	14103	14048	13848	14406	14198
Avg. district %	31.2%	31.1%	31.0%	30.7%	32.1%	31.7%

Table 2: Average votes obtained by the elected candidates under each method, and average percentage of votes obtained by elected candidates with respect to district votes.

TPP and TPP3 achieve a higher average than CCM, despite that this is a greedy mechanism: It just replaces candidates when necessary and in a local manner. One explanation for this is precisely the presence of locally top-voted candidates who are not elected in other methods and obtained a considerable number of votes. This is particularly relevant since, in addition to the property of plurality—and a representation threshold in the case of TPP3—these methods obtain the best results in terms of representation, followed by CCM and BPM. It is observed that the threshold decreases the average votes in this case. This behavior, however, is not a direct consequence of its application but rather depends on the instance. In this particular case, the negative effect over candidates of small lists who were not elected due to this threshold was more important than the positive effect over candidates of bigger lists.

4.3 Robustness

Another criterion we use to compare the methods is their robustness to small perturbations in the votes. To evaluate this aspect we conduct $n = 150$ simulations, and in each we multiply the votes obtained by each candidate by a normally distributed value with mean one and standard deviation 0.05, 0.1, and 0.2. We then compute the distribution of the number of seats transferred from one list to any other on each simulation starting from the original apportionment. Denoting the seats obtained by each list $\ell \in L$ in the original apportionment as x^0_ℓ, the seats obtained by each list $\ell \in L$ in simulation $i \in \{1, \dots, n\}$ as x^i_ℓ and the variable of interest as T^i, this variable is given by[9] $T^i = \frac{1}{2}\sum_{\ell \in L} |x^0_\ell - x^i_\ell|$. Figure 3 in the Appendix plots the distribution of this variable under each method and for each standard deviation considered. Since the effects of the perturbations tend to be compensated when considering more candidates, the local methods (CCM and BPM) have a greater mean and dispersion in the transference of seats, while three-proportional methods are consistently less sensitive to vote shocks. There is also an important difference in favor of methods with a threshold (TPM3 and TPP3), which can be explained by noting that smaller lists usually have fewer candidates, and therefore the perturbation of individual votes has a greater relative effect on the votes of the lists. A similar phenomenon is observed when studying deviations by gender.

4.4 One Person, One Vote?

As a final criterion, we compare the value of a cast vote under each method.[10] Under local methods (CCM and BPM), each vote counts only for electing the seats assigned to the corresponding district, and therefore its power can be measured as the number of seats per vote in that district. Figure 2 in the Appendix shows this indicator for each district, as a ratio of the global number of seats per vote. It is clear that the votes of people living in central districts are less powerful, as defined previously, than the votes of people living in extreme districts, reaching a factor of 5.93 between them. On the opposite, global methods without plurality ensure that, in terms of political representation, every vote is equally valuable. Global methods with plurality combine both features—global representation and election of locally top-voted candidates—so each vote is valuable in terms of national political distribution but has special relevance for the district where it is cast. This is related to the aforementioned trade-off between local and global representation, and the main observation here is that incorporating the different criteria as additional dimensions, rather than separating the election across them, may be a reasonable way to get closer to the widely known principle of *one person, one vote.*

[9]Note that since seat transfers are counted twice in the summation, we divide the expression by 2.

[10]Unlike the other subsections, in this one the votes of the lists without enough votes to obtain a seat under some method are considered.

5 DISCUSSION AND CONCLUSIONS

In this work, we proposed and studied different electoral methods based on multidimensional proportional representation. We use the Chilean Constitutional Convention election as a testing ground to analyze the outcomes of each method in terms of political representation, representativeness, robustness, and how well they accomplish the ideal of *one person, one vote.*

Regarding proportionality in political representation, we find an important trade-off between global and local representation, that is, local methods (CCM and BPM) generate global distortion in lists allocation, and global methods (TPM, TPM3, TPP, TPP3) generate district distortion. In this trade-off, the TPP presents a middle ground as it performs well in terms of a global allocation of seats to lists and it is the method based on three-proportionality that generates the least district distortion. This is due to the fact it elects the top-voted candidates in every district, and therefore these highly voted candidates are not excluded just because they run in small local projects, that are voted in their districts but not nationally. Then, the TPP method performs well in allocating seats to lists at the national level without generating much distortion in districts. When considering the average vote of elected candidates, the TPP method also exhibits the best performance. The threshold of 3% lowers the average votes obtained both in TPM and TPP, but this is not a direct consequence of its application. Then, the 3% threshold may be a reasonable property to include in a method that tends to elect candidates from too many small lists, in order to avoid this and favor the conformation of larger political coalitions.

The methods including a threshold of 3% (TPM3, TPP3) are the most robust to random shocks in votes, and this is precisely because the effect of shocks is less important in larger lists with various candidates across the country, where the perturbations tend to compensate between them, than in smaller lists with few candidates. A similar phenomenon occurs with local methods such as CCM and BPM. Since they run locally in each district, these methods are more sensitive to voting shocks as they consider a smaller amount of votes and candidates than if it were national, unlike methods based on three-proportionality.

Regarding voting power among districts, we observe that in some regions representatives are chosen with a much smaller amount of votes than those chosen in the center of the country. This representation problem is attenuated in global apportionment methods as the allocation of seats to lists is given by the national amount of votes a list gets, and then the apportionment is not completely restricted to district results as it is in local methods.

The presented methods can be valuable and feasible proposals for the election of representative bodies. Proportional or mixed-member proportional representation systems are two families of electoral systems widely used worldwide, and some countries also use biproportional apportionment methods [12, 13]. We show that relevant properties for an electoral system such as the representation threshold that favors the conformation of broader political projects, or the constraint of electing the most voted candidate in every district, can be incorporated in the design of methods based on multidimensional proportionality. In particular, the three-proportional method with plurality and threshold (TPP3) incorporates these properties and has a good performance in the various criteria proposed in the previous section.

Although multidimensional proportional methods may appear harder to understand for citizens at first glance, all methods incorporating additional constraints as gender balance suffer for this problem, and the concepts of proportionality, gender balance, and plurality directly imposed in the proposed methods are certainly intuitive in contrast with the somehow subtle corrections made in CCM. It is also important to mention that even though the ease of understanding constitutes a relevant element in terms of the legitimacy of the electoral process, a proper description and an appropriate performance of the mechanism do as well. Moreover, three-proportional methods are easily extended to more dimensions, for instance, to include ethnicity as a fourth dimension and allow ethic seats to be proportionally assigned across political, geographical, and gender distribution as well. We remark that methods based on multidimensional proportionality are computationally efficient, both theoretically and in practice. Furthermore the observed deviations from the prescribed marginals required to implement three-proportionality are in most cases zero or very small.

All these facts consolidate the possibility of thinking deeper on electoral methods incorporating several dimensions and constraints. This future work certainly requires an interdisciplinary approach in order to succeed in designing mechanisms able to better represent the complexity and diversity of modern societies. For example, this work does not account for the psychological effects of the new rules in the voters and parties' strategic behavior. Another direction concerns the relationship of the electoral rules with the legitimacy of the elected representative bodies. From a mathematical and computational perspective, this work shows that advanced optimization and algorithmic tools are valuable for designing and testing new electoral methods for increasingly complex societies, and opens the way for further approaches.

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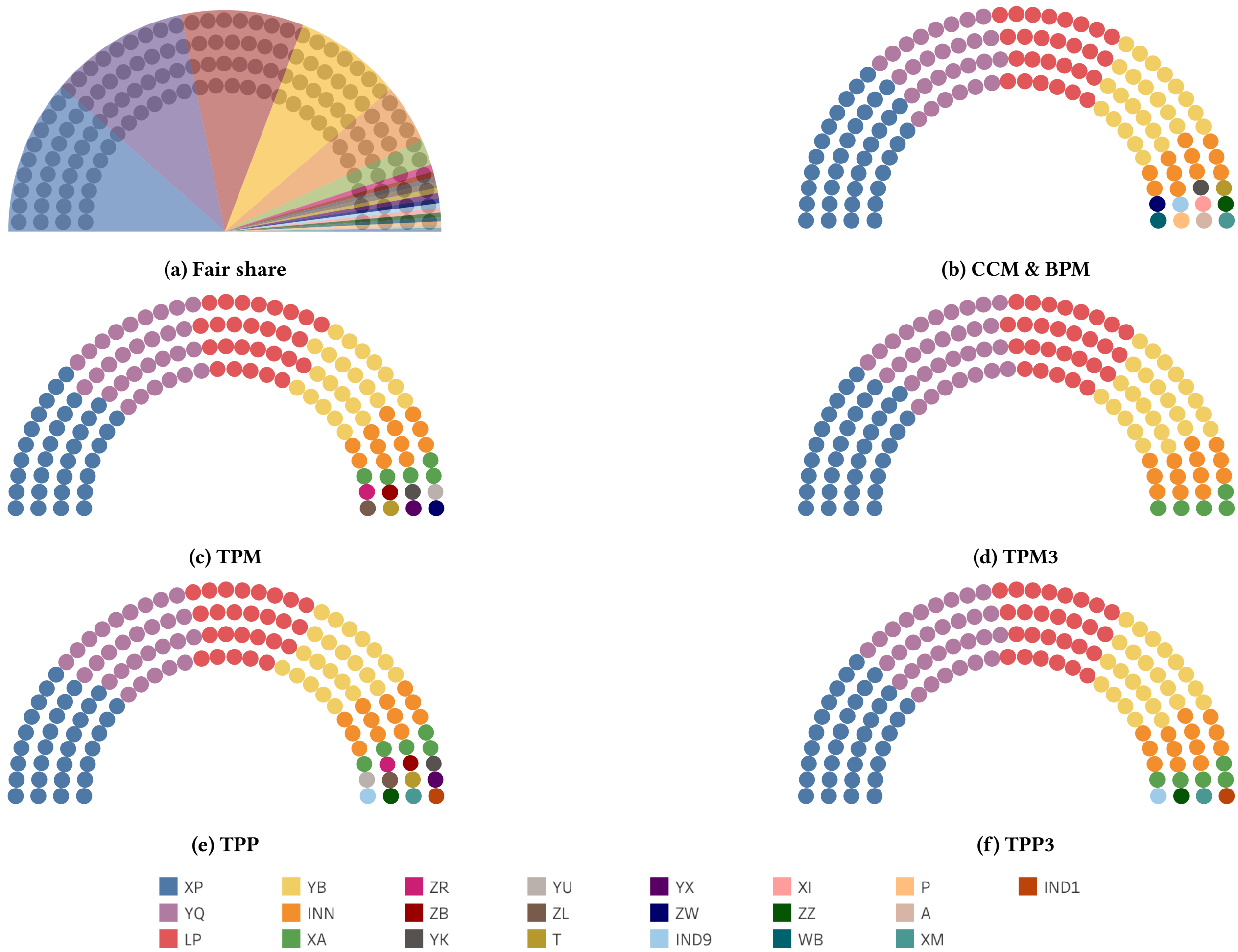
A OVERVIEW OF THE THEORETICAL TOOLS

We now give an overview of some of the technical ideas behind our work. Due to space constraints, in this appendix, we only highlight the main ideas of this approach, and defer to the full version the complete analysis. The TPM and TPM3 methods are based on the existential and algorithmic result by Cembrano et al. [8] in the context of three-dimensional proportional solutions with an additional pre-processing step to account for the 3% threshold to obtain representation. In the case of the TPP and TPP3 methods the picture changes since we need to guarantee the election of the most top-voted candidate in each district. To do this we incorporate a new constraint on the set of feasible allocations to the Cembrano et al's formulation. Then, a natural question is whether the existential and algorithmic result by Cembrano et al. remains valid in this setting. From the theoretical side, our contribution is to provide a positive answer to this question and therefore generalize that result for elections with pluralities.

In order to do this, we study an integer linear program similar to the one used in the three-dimensional proportional case without plurality, but now considering constraints that ensure the election of the top-voted candidate in each district. Apart from these constraints, the program ensures that its solution verifies the marginals and its objective function allows to obtain the new proportionality condition through a primal-dual analysis. We show that if the linear relaxation of this integer linear program is feasible, then a u-approximate three-proportional apportionment with plurality is guaranteed to exist whenever $1/(u_D+2)+1/(u_L+2)+1/(u_G+2) \leq 1$. Such an apportionment can be founded by first solving this linear program and checking whether the solution is integer or not. If it is integral, then it corresponds to a three-proportional apportionment with plurality. Otherwise, using the iterative rounding algorithm of Cembrano et al. [8] we can achieve a solution satisfying the new proportionality condition and deviating at most u_D from the marginals of each district, u_L from the marginals of each list, and u_G from the marginals of each gender.

B FIGURES



(a) Fair share

(b) CCM & BPM

(c) TPM

(d) TPM3

(e) TPP

(f) TPP3

Figure 1: Political representation obtained by list under each method and fair share apportionment for comparison.

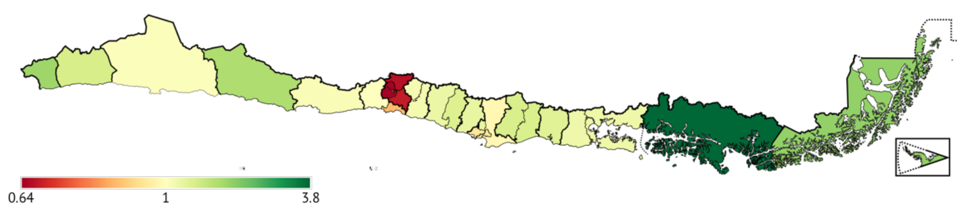
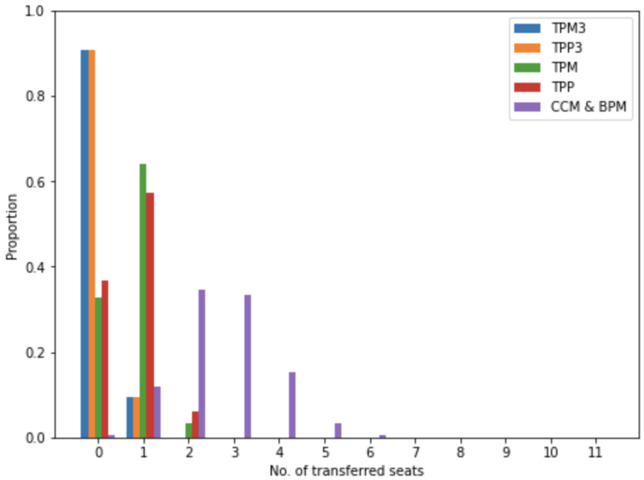
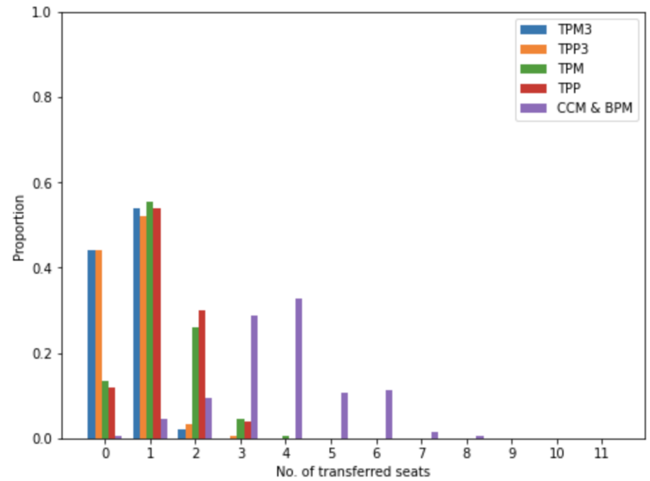


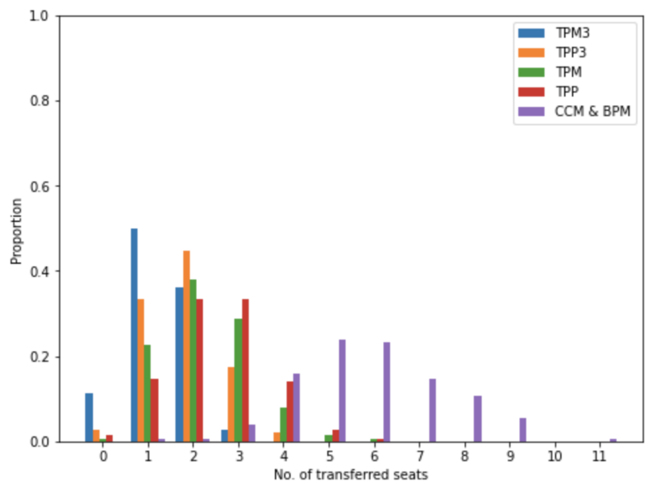
Figure 2: Vote power by district as ratio of global vote power.



(a) 5% Std. Dev. Shock



(b) 10% Std. Dev. Shock



(c) 20% Std. Dev. Shock

Figure 3: Number of transferred seats distribution under each system and for different standard deviation for the normal distribution of votes shocks.