Self-clique graphs and matrix permutations

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Abstract

The *clique graph* of a graph is the intersection graph of its (maximal) cliques. A graph is *self-clique* when it is isomorphic with its clique graph, and is *clique-Helly* when its cliques satisfy the Helly property. We prove that a graph is clique-Helly and self-clique if and only if it admits a *quasi-symmetric* clique matrix, that is, a clique matrix whose families of row and column vectors are identical. We also give a characterization of such graphs in terms of vertex-clique duality.

We describe new classes of self-clique and 2-self-clique graphs. Further, we consider some problems on permuted matrices (matrices obtained by permuting the rows and/or columns of a given matrix). In particular, we prove that deciding whether a (0,1)-matrix admits a symmetric (quasi-symmetric) permuted matrix is graph (hypergraph) isomorphism complete.

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1 Introduction

Let G be a finite connected graph, with vertex set $V(G)$ and edge set $E(G)$. A (maximal) *clique* of G is a maximal set of pairwise adjacent vertices of G. We denote by $\mathcal{C}(G)$ the collection of cliques of G. The graph G is *clique-Helly* when $\mathcal{C}(G)$ satisfies the Helly property, that is, when any subcollection of pairwise intersecting cliques contains a common vertex. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G (that is, the graph whose vertices are the cliques of G, two being adjacent if and only if they intersect). Define $K^p(G)$ recursively by $K^0(G) := G$ and $K^p(G) := K(K^{p-1}(G))$, $p \geq 1$. A graph G is *self-clique* if $K(G) \cong G$. More generally, if there is a positive integer p such that that $K^p(G) \cong G$, and p is the least such integer, the graph G is called *p-self-clique*.

Although the study of clique graphs was begun over 30 years ago [8,13], very little is known about self-clique graphs. Indeed, no good characterization of this class of graphs is known, even when restricted to clique-Helly graphs. On the other hand, clique-Helly graphs which are either self-clique or 2-self-clique do admit a simple characterization, obtained by Escalante [5]. Its statement involves the concept of a dominated vertex.

For $v \in V(G)$, denote by $N(v)$ the set of neighbours of v and by $d(v)$ the degree of v. Let $N[v] := \{v\} \cup N(v)$. The vertex v is *dominated* by vertex w if $N[v] \subseteq N[w]$.

Theorem 1 [5]. *Let G be a clique-Helly graph. Then G is self-clique or 2-self-clique if and only if G has no dominated vertices.*

The following is a direct consequence. We denote by $\delta(G)$ the minimum vertex degree of G.

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Corollary 2 [5]. *If G is triangle-free and $\delta(G) \geq 2$, then G is 2-self-clique, unless G is a cycle, in which case it is self-clique.*

The latter result was extended by Chia [3], who characterized the self-clique graphs all of whose cliques have size two, except for precisely one clique. Balakrishnan and Paulraja [1] described further examples of self-clique graphs. The class was studied also by Escalante and Toft [6].

Let G be a graph, $v_1, v_2, \ldots, v_n$ a list of its vertices and $C_1, C_2, \ldots, C_m$ a list of its cliques. The $(0,1)$-matrix $A = (a_{ij})$, where $a_{ij} = 1$ if and only if vertex v_i belongs to clique C_j is called a *clique matrix* of G. In this paper, we characterize the self-clique graphs which are clique-Helly in terms of their clique matrices. Further, we describe new classes of self-clique graphs. Finally, self-clique and 2-self-clique graphs motivate our study of computational aspects of certain types of permuted matrices.

We remark in passing that there exists a polynomial-time algorithm for recognizing clique matrices, due to P.C. Gilmore (see Cornuéjols [4]), and also one for recognizing clique-Helly graphs, due to Szwarcfiter [15].

Let $A = (a_{ij})$ be a matrix. A matrix $A' = (a'_{ij})$ derived from A by permuting the columns of A is called a *permuted matrix* of A. Thus $A' = AP$, where P is a permutation matrix, or equivalently $a'_{ij} = a_{ip(j)}$, where p is a permutation of the columns of A; we refer to A' as the permuted matrix of A *relative to* P (or p). A *doubly permuted matrix* of A is a matrix QAP, where P and Q are permutation matrices, in other words, a matrix derived from A by independent permutations, p and q, of its columns and rows, respectively. A matrix is *quasi-symmetric* if its families of row and column vectors are identical; in particular, every symmetric matrix is quasi-symmetric.

The plan of the paper is as follows. The next section describes characterizations of clique-Helly self-clique graphs in terms of permuted matrices of their clique matrices, and also in terms of the concept of vertex-clique duality. New classes of self-clique graphs are presented in Section 3. In Section 4, we formulate generalizations of Corollary 2, leading to new classes of 2-self-clique graphs. In Section 5, we prove that the problem of recognizing whether a given matrix admits a symmetric permuted matrix is graph isomorphism complete, and that recognizing whether a matrix admits a quasi-symmetric permuted matrix is hypergraph isomorphism complete.

Certain other computational permuted matrix problems have been studied by various authors, among them the problem of minimizing bandwidth. This problem has been proved to be NP-hard [11], remaining so even for adjacency matrices of trees [7]. However, if the value of the bandwidth is a fixed integer, the problem can solved in polynomial time [14]. Finally, we note that permuted matrices with forbidden configurations are considered in [9].

2 Characterizing clique-Helly self-clique graphs

In this section, we characterize the clique-Helly self-clique graphs.

For a graph $G = (V, E)$, and a vertex $v \in V$, we denote by $\mathcal{C}(v)$ the set of cliques of G containing v.

Proposition 3 *Let G be a clique-Helly graph and let $\mathcal{C}$ be a maximal set of pairwise intersecting cliques of G. Then $\mathcal{C} = \mathcal{C}(v)$, for some $v \in V$. If, moreover, G is self-clique, the maximal sets of pairwise intersecting cliques of G are precisely the sets $\mathcal{C}(v)$, $v \in V$.*

Proof By the Helly property, the cliques in $\mathcal{C}$ have a common vertex v. Thus $\mathcal{C} \subseteq \mathcal{C}(v)$. Since $\mathcal{C}$ is maximal, $\mathcal{C} = \mathcal{C}(v)$.

Suppose now that G is also self-clique. If $\mathcal{C}(v)$ is not maximal, $\mathcal{C}(v) \subset \mathcal{C}(w)$ for some $w \in V$. But then v is dominated by w, contradicting Theorem 1. It follows that the sets $\mathcal{C}(v)$, $v \in V$, are precisely the maximal sets of pairwise intersecting cliques of G. ∎

A *permutation matrix* is a $(0, 1)$-matrix with precisely one 1 in each row and each column.

Proposition 4 *For a matrix A, the following conditions are equivalent:*

(i) *A is quasi-symmetric;*
(ii) *$A = A^t P$ for some permutation matrix P;*
(iii) *$A = PA^t$ for some permutation matrix P.*

Proof A matrix $A = (a_{ij})$ is quasi-symmetric if and only if there is a permutation p of $\{1, 2, \ldots, n\}$ such that $a_{ij} = a_{jp(i)}$ for all i, j, $1 \leq i, j \leq n$, that is,

$$A = (a_{ij}) = (a_{jp(i)}) = A^t P$$

for some permutation matrix P. Thus (i) and (ii) are equivalent.

Also

$$A = A^t P \Leftrightarrow A^t = (A^t P)^t = P^t A = P^{-1} A \Leftrightarrow A = PA^t.$$

Thus (ii) and (iii) are equivalent. ∎

Let G be a self-clique graph, with vertices $v_1, v_2, \ldots, v_n$. Then there is an ordering $C_1, C_2, \ldots, C_n$ of the cliques of G such that

$$v_i v_j \in E(G) \quad \text{if and only if} \quad C_i \cap C_j \neq \emptyset.$$

We shall refer to the clique matrix of G with respect to this ordering as a *self-clique matrix* of G.

Theorem 5 *A graph is clique-Helly and self-clique if and only if it has a quasi-symmetric clique matrix.*

Proof Let G be a clique-Helly self-clique graph. Consider a self-clique matrix A of G. Set $J(i) := \{j : a_{ij} = 1\}$, $1 \leq i \leq n$. By Proposition 3, $\mathcal{C}(v_i) = \{C_j : j \in J(i)\}$ is a maximal pairwise intersecting set of cliques of G. Because A is a self-clique matrix of G, the set $\{v_j : j \in J(i)\}$ is a maximal clique of G, say $C_{p(i)}$. Thus the $(0,1)$-vectors defined by row i and column $p(i)$ of A are identical. Moreover, p is a permutation of $\{1, 2, \ldots, n\}$, because the sets $\mathcal{C}(v_i)$, $1 \leq i \leq n$, are distinct, by Proposition 3. Thus A is quasi-symmetric.

Conversely, let G be a graph with a quasi-symmetric clique matrix $A = (a_{ij})$, where $a_{ij} = a_{jp(i)}$. We prove, first, that $v_i v_j \in E(G)$ if and only if $C_{p(i)} \cap C_{p(j)} \neq \emptyset$. Let $v_i v_j \in E(G)$. There is a clique C_k containing both v_i and v_j. Thus $a_{ik} = a_{jk} = 1$, implying that $a_{kp(i)} = a_{kp(j)} = 1$. Because A is a clique matrix of G, $v_k \in C_{p(i)} \cap C_{p(j)}$. Consequently, $C_{p(i)} \cap C_{p(j)} \neq \emptyset$. Conversely, suppose that $C_{p(i)} \cap C_{p(j)} \neq \emptyset$. Then there exists a vertex $v_k \in C_{p(i)} \cap C_{p(j)}$, that is, $a_{kp(i)} = a_{kp(j)} = 1$, so $a_{ik} = a_{jk} = 1$. Hence $v_i, v_j \in C_k$, implying that $v_i v_j \in E(G)$, as required. Consequently, G is self-clique.

It remains to show that G is clique-Helly. Let $\mathcal{C}$ be a set of pairwise intersecting cliques of G and set $S := \{v_i : C_{p(i)} \in \mathcal{C}\}$. Because G is self-clique, $v_i v_j \in E(G)$ if and only if $C_{p(i)} \cap C_{p(j)} \neq \emptyset$. Consequently, S is contained in a clique C_k of G. Because A is a clique matrix, $a_{ik} = 1$, for all i such that $v_i \in S$. Thus $a_{kp(i)} = 1$ for all i such that $v_i \in S$, in other words, $v_k \in C_{p(i)}$ for all $C_{p(i)} \in \mathcal{C}$, so G is clique-Helly. ■

In view of Theorem 5, one is led to ask whether every clique-Helly self-clique graph has a *symmetric* clique matrix. This, however, is not so. Figure 1 shows a graph with 12 vertices. This graph is clique-Helly and self-clique, but an exhaustive computer check shows that it has no symmetric clique matrix. It is, moreover, a smallest graph with these properties.

We now give a second characterization of clique-Helly self-clique graphs. It involves the notion of vertex-clique duality. Let G be a graph with the same number of vertices and cliques. A *vertex-clique duality* of G is a bijection ϕ which maps vertices to cliques and cliques to vertices in such a way that,

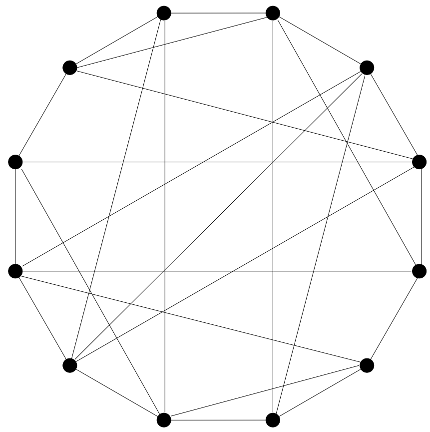


Fig. 1. A clique-Helly self-clique graph with no symmetric clique matrix

for $v \in V$ and $C \in \mathcal{C}$, $v \in C$ if and only if $\phi(C) \in \phi(v)$. This notion was introduced under the name *self-duality* by Balconi [2].

Theorem 6 *A graph is clique-Helly and self-clique if and only if it admits a vertex-clique duality.*

Proof By Theorem 5 and Proposition 4, a graph G is clique-Helly and self-clique if and only if it has a clique matrix A such that $A = A^tP$ for some permutation matrix $P = (p_{ij})$. Let $V(G) := \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{C}(G) := \{C_1, C_2, \ldots, C_n\}$, where $a_{ij} = 1$ if and only if $v_i \in C_j$. Define the mapping $\phi : V \cup \mathcal{C} \to V \cup \mathcal{C}$ by $\phi(v_i) = C_i$, $1 \leq i \leq n$, and $\phi(C_j) = v_k$, where k is the index for which $p_{kj} = 1$, $1 \leq j \leq n$.

Suppose that $v_i \in C_j$. Then $a_{ij} = 1$, which implies, in view of the matrix equation $A = A^tP$, that $a_{ki} = p_{kj} = 1$, for some k. Thus $v_k \in C_i$ and $p_{kj} = 1$, for some k. Equivalently, by the definition of ϕ, $\phi(C_j) \in \phi(v_i)$. These implications are, moreover, reversible. The mapping ϕ is thus a vertex-clique duality of G.

Conversely, given a vertex-clique duality ϕ of G, where we assume, by permuting the cliques of G if necessary, that $\phi(v_i) = C_i$, $1 \leq i \leq n$, the clique matrix A of G satisfies the equation $A = A^tP$ for a suitable permutation matrix P. ■

3 Classes of self-clique graphs

In this section, we describe two new classes of clique-Helly self-clique graphs. Examples of connected self-clique graphs which are not clique-Helly were given by Escalante [5], who also described infinite families of disconnected non-clique-Helly p-self-clique graphs for arbitrary $p > 1$.

Let G be a graph, and let u and v be two vertices of G. A *uv-geodesic* is a shortest uv-path in G. Note that, in a graph of girth greater than 2ℓ, any two vertices at distance ℓ are connected by a unique geodesic. We denote by $D_k(v)$ the set of vertices of G at distance at most k from v.

Proposition 7 *For any graph G and any vertex v of G, $D_k(v)$ is a clique of G^{2k}. Furthermore, if $\delta(G) \geq 2$, then*

(a) if $\mathrm{girth}(G) \geq 2k+2$, the subgraph of G induced by $D_k(v)$ is a subtree $T_k(v)$ with at least two leaves, all of which are at distance k from v, and the cliques $D_k(v)$, $v \in V$, are distinct;

(b) if $\mathrm{girth}(G) \geq 4k+2$, the cliques $D_k(v)$, $v \in V$, are maximal cliques of G^{2k};

(c) if $\mathrm{girth}(G) \geq 6k+1$, the cliques $D_k(v)$, $v \in V$, are the only maximal cliques of G^{2k}.

Proof Since the distance in G between v and any other vertex of $D_k(v)$ is at most k, the distance between any two vertices of $D_k(v)$ is at most $2k$. Therefore $D_k(v)$ is a clique of G^{2k}. Assume now that $\delta(G) \geq 2$.

(a). The subgraph $T_k(v)$ of G induced by $D_k(v)$ is clearly connected. Moreover, if $\mathrm{girth}(G) \geq 2k+2$, $T_k(v)$ is acyclic, hence a tree. Because $\delta(G) \geq 2$, $T_k(v)$ has at least two leaves. Furthermore, since $\delta(G) \geq 2$ and $\mathrm{girth}(G) \geq 2k+2$, all maximal paths of G starting at v are of length at least $2k+1$. This implies that the leaves of $T_k(v)$ are at distance k from v. If $D_k(u) = D_k(v)$, where $u \neq v$, then $u \in D_k(v)$. Because $T_k(v)$ has at least two leaves, the uv-path in $T_k(v)$ can be extended to a path in $T_k(v)$ of length $k+1$. Let w be the terminal vertex of this path. Since $\mathrm{girth}(G) \geq 2k+2$, the vertex w is at distance $k+1$ from u. Thus $w \in D_k(v) \setminus D_k(u)$, a contradiction.

(b). Suppose, now, that $\mathrm{girth}(G) \geq 4k+2$. If $D_k(v)$ is not a maximal clique of G^{2k}, some vertex $u \notin D_k(v)$ is adjacent in G^{2k} to every vertex of $D_k(v)$. In G, the distance between u and v is greater than k and at most $2k$. Because $T_k(v)$ has at least two leaves and $\mathrm{girth}(G) \geq 4k+2$, some vertex w of $T_k(v)$ is at distance $2k+1$ from u, contradicting the assumption that $uw \in E(G^{2k})$.

(c). Suppose, finally, that $\mathrm{girth}(G) \geq 6k+1$. Let C be a maximal clique of G^{2k}. There exist vertices $u, w \in C$ whose distance in G is $2k$, for otherwise any neighbour in G of a vertex in C could be adjoined to C, thereby yielding a clique of G^{2k} containing C. Let v be the midpoint of the unique uw-geodesic P. We claim that $C = D_k(v)$. Let $x \in C$, let Q be the unique xu-geodesic and R the unique xw-geodesic. Since $\mathrm{girth}(G) \geq 6k+1$ and the lengths of P, Q and R sum to no more than $6k$, $P \cup Q \cup R$ is acyclic, hence a subtree of G. It follows that $x \in D_k(v)$, and so $C \subseteq D_k(v)$. Since C and $D_k(v)$ are maximal cliques, $C = D_k(v)$. ■

Theorem 8 *Let G be a graph with $\delta(G) \geq 2$ and $\mathrm{girth}(G) \geq 6k+1$, where $k \geq 1$. Then G^{2k} is a clique-Helly self-clique graph.*

Proof By Proposition 7, the maximal cliques of G^{2k} are the sets $D_k(v)$, $v \in V$. We show that the involution $\phi : V(G^{2k}) \cup \mathcal{C}(G^{2k}) \to V(G^{2k}) \cup \mathcal{C}(G^{2k})$ defined by $\phi(v) = D_k(v)$ and $\phi(D_k(v)) = v$, $v \in V$, is a vertex-clique duality of G^{2k}:

$$u \in D_k(v) \Leftrightarrow d_G(u, v) \leq k \Leftrightarrow v \in D_k(u) \Leftrightarrow \phi(D_k(v)) \in \phi(u).$$

It follows from Theorem 6 that G^{2k} is a clique-Helly self-clique graph. ■

The above theorem is best possible in the following sense: if either $\delta(G) < 2$ or $g(G) < 6k + 1$, then G^{2k} is not necessarily a self-clique graph. A graph G formed by a C_7, together with an additional vertex adjacent to exactly one vertex of the cycle, is an example where the degree condition of the theorem fails, while the girth condition is satisfied for $k = 1$. However G^2 is not self-clique. On the other hand, there is no graph G of minimum degree at least two and girth $6k$ such that G^{2k} is self-clique, because G^{2k} then has more maximal cliques than vertices: apart from the (distinct) maximal cliques $D_k(v)$, $v \in V$, G^{2k} has 3-vertex cliques consisting of equally spaced vertices on the circuits of G of length $6k$.

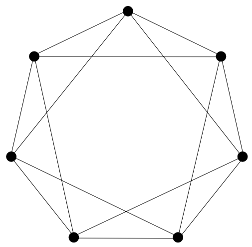


Fig. 2. A self-clique graph derived from Theorem 8.

In the remainder of this section, we describe a second class of self-clique graphs. It is based on the following expansion operation.

Let G be a graph and S a set of vertices of G. The *expansion of G relative to S*, is the graph H obtained from G by iteratively replacing each vertex $v \in S$ by a clique $C(v)$ of size $d_G(v)$ and making each $w \in C(v)$ adjacent in H to exactly one neighbour of v in G, these neighbours being distinct. We call $C(v)$ the *expanded clique* at v. For example, the graph of Figure 3(b) is the expansion of that of Figure 3(a), relative to $\{y_1, y_2, y_3\}$.

The following definition is useful. Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. The *reduced adjacency matrix* of G is the $m \times n$ $(0, 1)$-matrix $B = (b_{ij})$, where $b_{ij} = 1$ if and only if $x_i y_j \in E(G)$.

Theorem 9 *Let G be a bipartite graph, with bipartition (X, Y), admitting a symmetric reduced adjacency matrix. Then the expansion of G relative to Y is a clique-Helly self-clique graph.*

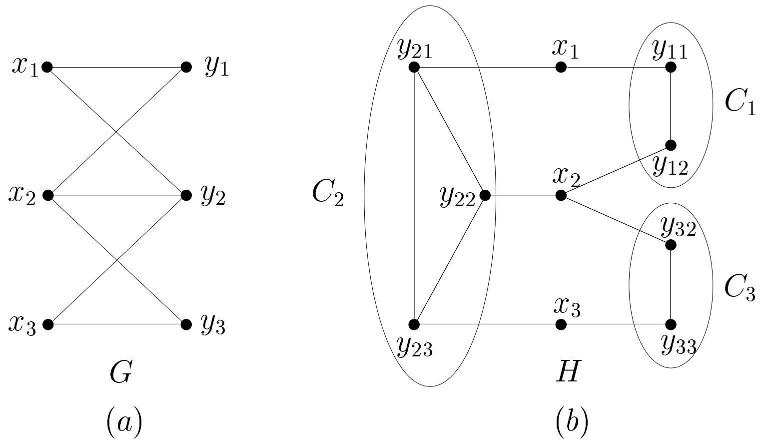


Fig. 3. (a) A graph satisfying the hypothesis of Theorem 9 and (b) its expansion

Proof First, we show that the expansion G' of G relative to Y is clique-Helly. A graph which is not clique-Helly contains at least four pairwise intersecting cliques, each of size at least three. However, each clique of G' of size at least three intersects only cliques of size two. Consequently, G' is clique-Helly.

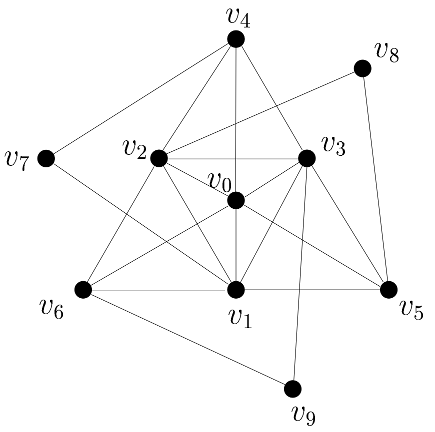
Let $B = (b_{ij})$ be the symmetric reduced adjacency matrix of G, and let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be the vertices of X and Y, in the ordering of the rows and columns of B, respectively. Set $m := |E(G)|$. Denote by H the expansion of G relative to Y, and by $C_i := C(y_i)$ the expanded clique of $y_i \in Y$, $1 \leq i \leq n$. For $x_i y_j \in E(G)$ (or, equivalently, for $x_j y_i \in E(G)$, because the matrix B is symmetric) denote by y_{ij} the vertex of C_i adjacent to x_j in H, and by C_{ij} the clique $\{y_{ij}, x_j\}$. The graph H contains exactly $n + m$ cliques, namely the n cliques C_i, $1 \leq i \leq n$, and the m cliques C_{ij}, where i, j are such that $x_i y_j \in E(G)$.

Consider the mapping $f : V(H) \to \mathcal{C}(H)$ defined by

$$\begin{aligned} f(x_i) &= C_i, \ 1 \leq i \leq n, \\ f(y_{ij}) &= C_{ji}, \ x_i y_j \in E(G), \ 1 \leq i, j \leq n. \end{aligned}$$

Let $uv \in E(H)$. Then either $u = y_{ij}$ and $v = x_j$ for some i, j such that $x_i y_j \in E(G)$, or $u = y_{ij}$ and $v = y_{ik}$, for some i, j, k such that $x_i y_j, x_i y_k \in E(G)$. In the former case, $f(u) = C_{ji}$ and $f(v) = C_j$, so $f(u) \cap f(v) = \{y_{ji}\} \neq \emptyset$. In the latter case, $f(u) = C_{ji}$ and $f(v) = C_{ki}$, so $f(u) \cap f(v) = \{x_i\} \neq \emptyset$. Conversely, $f(u) \cap f(v) = \emptyset$ whenever $uv \notin E(H)$. ■

It turns out that the self-clique graphs described in this section are not only clique-Helly but have the stronger property of being *hereditary clique-Helly*, that is, all their induced subgraphs are clique-Helly. This can be shown by applying a characterization of hereditary clique-Helly graphs due to Prisner [12]. However, there do exist self-clique graphs which are clique-Helly but not hereditary clique-Helly. An example is shown in Figure 4, together with its (symmetric) clique matrix.

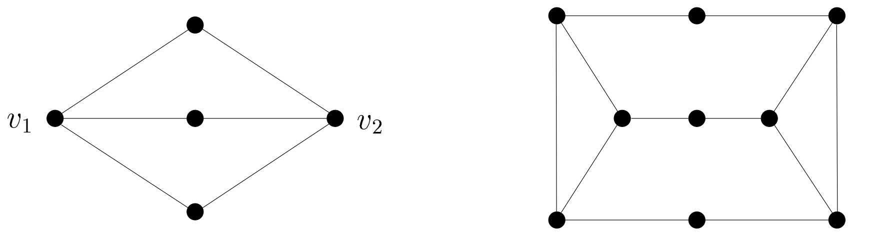


	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
v_0	1	1	1	1	0	0	0	0	0	0
v_1	1	0	1	1	1	0	0	0	0	0
v_2	1	1	0	1	0	1	0	0	0	0
v_3	1	1	1	0	0	0	1	0	0	0
v_4	0	1	0	0	0	0	0	1	0	0
v_5	0	0	1	0	0	0	0	0	1	0
v_6	0	0	0	1	0	0	0	0	0	1
v_7	0	0	0	0	1	0	0	1	0	0
v_8	0	0	0	0	0	1	0	0	1	0
v_9	0	0	0	0	0	0	1	0	0	1

Fig. 4. A self-clique graph which is clique-Helly but not hereditary clique-Helly, and its clique matrix

4 Classes of 2-self-clique graphs

In this section, we formulate two propositions which generalize Corollary 2, leading to broader classes of 2-self-clique graphs. In each case, we show that the graphs under consideration are clique-Helly and have no dominated vertices. By Theorem 1, they are therefore either self-clique or 2-self-clique. Counting cliques serves to eliminate the former eventuality.



$S = \{v_1, v_2\}$

Fig. 5. A graph satisfying the hypothesis of Theorem 10, and its expansion

Theorem 10 *Let G be a graph with $\delta(G) \geq 2$ and let S be a set of vertices of G such that*

(i) $G - S$ is triangle-free;
(ii) $|E(G)| - |V(G)| \neq \sum_{v \in S}(d_G(v) - 2)$.

Then the expansion of G relative to S is a 2-self-clique graph.

Proof First, we note that, as in the proof of Theorem 9, the expansion G' of G relative to S is clique-Helly. It thus satisfies the hypothesis of Theorem 1.

Next, we prove that G' has no dominated vertices. The edges of G' are of four types:

(1) edges with ends in the same expanded clique;
(2) edges with ends in distinct expanded cliques;
(3) edges with one end in an expanded clique and the other in $V \setminus S$;
(4) edges of $G - S$.

Note that, since $\delta(G) \geq 2$, each expanded clique has at least two vertices. For a vertex v in an expanded clique, we shall refer to the neighbours of v in the clique as its *internal* neighbours, and the one remaining neighbour as its *external* neighbour. By the construction of G', no two vertices of the same expanded clique have a common external neighbour.

Let vw be an edge of G'. In case (1), v and w have distinct external neighbours, and in case (2), v and w have distinct internal neighbours. In case (3), with $w \in V \setminus S$, v has an internal neighbour v' and w has a neighbour w' other than v, since $\delta(G) \geq 2$; moreover $v' \neq w'$. In case (4), both v and w have second neighbours, since $\delta(G) \geq 2$. These neighbours are necessarily distinct: no vertex of an expanded clique can have both v and w as external neighbours; likewise, no vertex of the triangle-free subgraph $G-S$ can be adjacent to both v and w. Consequently, G' has no dominated vertices. From Theorem 1, we conclude that G' is either self-clique or 2-self-clique.

We now compute the number $|\mathcal{C}(G')|$ of cliques of G'. The cliques of G' are of three types:

(1) cliques of $G - S$;
(2) expanded cliques;
(3) cliques of G' consisting of a vertex of an expanded clique and its external neighbour.

The cliques of type 1 correspond to edges of G, because $G-S$ is triangle-free. The cliques of type 3 also correspond to edges of G. There are $|E(G)|$ cliques of these two types. In addition, there are $|S|$ cliques of type 2. Consequently, $|\mathcal{C}(G')| = |E(G)|+|S|$. The number of vertices of G' is $|V(G')| = |V(G)|-|S|+\sum_{v\in S} d_G(v)$. Condition (ii) of the hypothesis implies that $|\mathcal{C}(G')| \neq |V(G')|$; in particular, G' is not self-clique. We conclude that G' is 2-self-clique. ■

When $S = \emptyset$, the above theorem reduces to Corollary 2. The next result is another generalization of Corollary 2.

Theorem 11 *Let G be a clique-Helly graph which has a spanning triangle-free subgraph H of minimum degree at least two, each of whose components is an induced subgraph of G and intersects each component of $G \setminus E(H)$ in at most one vertex. Then G is 2-self-clique, unless G consists of a set of disjoint cycles, in which case G is self-clique. In particular, the cartesian product $F \square H$ of a nonempty clique-Helly graph F with a triangle-free graph H of minimum degree at least two is 2-self-clique.*

Proof First, we show that G has no dominated vertices. Let $vw \in E(G)$. If $vw \in E(H)$, then v does not dominate w because H is an induced triangle-free subgraph of G and $\delta(H) \geq 2$. Suppose, then, that $vw \in E(G) \setminus E(H)$. Then v and w belong to distinct components of H. Let C be the component of $G \setminus E(H)$ containing the edge vw. In this case, too, v cannot dominate w because C intersects the component of H containing w in just one vertex, namely w, so the neighbours of w in this component are not neighbours of v. Consequently, G has no dominated vertices. Since G is clique-Helly, G is either self-clique or 2-self-clique, by Theorem 1.

We now compute the number of cliques of G and compare it with the number of vertices. Because each component of H intersects each component of $G \setminus E(H)$ in at most one vertex, $|\mathcal{C}(G)| = |\mathcal{C}(H)| + |\mathcal{C}(G \setminus E(H))|$. In addition, $|\mathcal{C}(H))| = |E(H)| \geq |V(G)|$ (equality holding only if H consists of a set of disjoint cycles). The result follows. ■

A simple example of a 2-self-clique graph obtained from Theorem 11 is the graph $C_3 \square C_4$, shown in Figure 6.

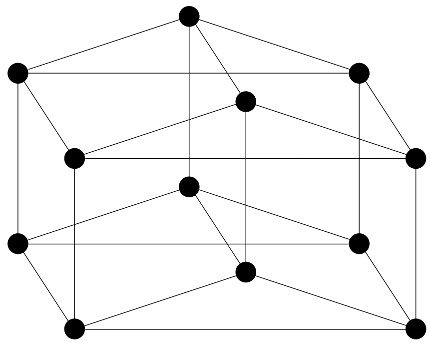


Fig. 6. The 2-self-clique graph $C_3 \square C_4$.

5 Decision problems for permuted matrices

In this section, we examine the complexity of two permuted matrix problems. Given a matrix A, the problems consist of deciding whether or not A admits (i) a symmetric permuted matrix, and (ii) a quasi-symmetric permuted matrix. These problems are motivated by Theorem 5.

The following theorem relates the complexity of deciding whether a $(0,1)$-matrix admits a symmetric permuted matrix to the graph isomorphism problem.

Theorem 12 *The problem of deciding whether a $(0,1)$-matrix admits a symmetric permuted matrix is graph isomorphism complete.*

Proof Let $A = (a_{ij})$ be a $(0,1)$-matrix. We can regard A as a reduced adjacency matrix of a bipartite graph G with bipartition (X,Y); that is, $x_i y_j \in E(G)$ if and only if $a_{ij} = 1$. First, we prove that A admits a symmetric permuted matrix if and only if G admits an involutory automorphism switching its bipartition (X,Y).

If the permutation p^{-1} of the columns of A results in a symmetric matrix $B = (b_{ij})$, then $a_{ip(j)} = b_{ij} = b_{ji} = a_{jp(i)}$, $1 \le i,j \le n$. The bijection $x_i \leftrightarrow y_{p(i)}$ is thus an automorphism of G switching its bipartition (X,Y):

$$x_i y_{p(j)} \in E(G) \Leftrightarrow a_{ip(j)} = 1 \Leftrightarrow a_{jp(i)} = 1 \Leftrightarrow x_j y_{p(i)} \in E(G) \Leftrightarrow y_{p(i)} x_j \in E(G).$$

Conversely, if there is an involutory automorphism of G switching its bipartition (X,Y), there is a permutation p such that $x_i \leftrightarrow y_{p(i)}$ is an automorphism of G. Therefore

$$a_{ip(j)} = 1 \Leftrightarrow x_i y_{p(j)} \in E(G) \Leftrightarrow y_{p(i)} x_j \in E(G) \Leftrightarrow x_j y_{p(i)} \in E(G) \Leftrightarrow a_{jp(i)} = 1,$$

and so the permutation p^{-1} of the columns of A results in a symmetric matrix.

We now show that there exists a polynomial-time algorithm for deciding isomorphism between arbitrary graphs if only if there is a polynomial-time algorithm to decide whether a bipartite graph admits an involutory automorphism switching its bipartition.

Suppose that we have an algorithm to decide whether a bipartite graph $B(X,Y)$ has an involutory automorphism which interchanges X and Y. Let G and H be two connected graphs of maximum degree at least three. Subdivide each edge of G and H by a vertex, call the resulting graphs G' and H' and denote the sets of subdividing vertices by $S(G)$ and $S(H)$, respectively. Then $G' \cup H'$ is a bipartite graph $B(X,Y)$, where $X = V(G) \cup S(H)$ and $Y = S(G) \cup V(H)$. The bipartite graph B has an involutory automorphism f which interchanges X and Y if and only if f maps $V(G)$ onto $V(H)$ and $S(G)$ onto $S(H)$ and thus induces an isomorphism of G and H.

Conversely, suppose that we have an algorithm to decide whether two graphs G and H are isomorphic. Let $B(X,Y)$ be a connected bipartite graph. Construct graphs G and H by adding a complete graph on X and Y, respectively. Then G and H are isomorphic if only if $B(X,Y)$ has an involutory automorphism which interchanges X and Y. ■

The next theorem considers quasi-symmetric matrices.

Theorem 13 *The problem of deciding whether a* $(0,1)$*-matrix admits a quasi-symmetric permuted matrix is hypergraph isomorphism complete.*

Proof Let $A = (a_{ij})$ be a $(0,1)$-matrix. Let $H(A)$ be the hypergraph whose vertex-edge incidence matrix is A. Clearly, the transpose A^t of A is precisely the incidence matrix of the dual H^* of H. Suppose that A admits a quasi-symmetric permuted matrix A'. It follows readily from Proposition 4 that A^t is a doubly permuted matrix of A. Consequently, H and H^* are isomorphic. The converse is established similarly. We can thus assert that A admits a quasi-symmetric permuted matrix if and only if the hypergraph $H(A)$ is isomorphic to its dual, that is, is self-dual.

We now examine the complexity of recognizing self-dual hypergraphs. Suppose that we have an algorithm to decide whether a hypergraph is self-dual. Consider two connected hypergraphs F and G. We wish to decide if these hypergraphs are isomorphic. Determine first whether they are self-dual. If one is self-dual and the other is not, they are not isomorphic. If one has an edge through all vertices and the other does not, they are not isomorphic. If F and G are self-dual and neither has an edge through all vertices, we add such an edge to each of them; if F and G are self-dual and both have an edge through all vertices, we delete this edge from each of them; if neither F nor G is self-dual, we do nothing. In all three cases, we denote the resulting hypergraphs by $\hat{F}$ and $\hat{G}$. These hypergraphs are not self-dual. Let H be the disjoint union of $\hat{F}$ and the dual of $\hat{G}$. Then H is self-dual if only if $\hat{F}$ and $\hat{G}$ are isomorphic, and this is so if and only if F and G are isomorphic. This proves that the problem of recognizing $(0,1)$-matrices admitting quasi-symmetric permuted matrices is hypergraph isomorphism complete. ∎

We remark that the general hypergraph isomorphism decision problem can be solved in time $O(c^n)$, where c is a constant and n is the number of vertices of the hypergraph [10]. We note, too, that Theorem 13 still leaves open the complexity status of recognizing clique-Helly self-clique graphs, because clique matrices are a special class of $(0,1)$-matrices.

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graphs constructed in Section 3 are, in fact, hereditary clique-Helly.

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