The intersection between some subclasses of circular-arc and circle graphs

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Abstract

The intersection graph of a family of arcs on a circle is called a circular-arc graph. This class of graphs admits some interesting subclasses: proper circular-arc graphs, unit circular-arc graphs, Helly circular-arc graphs and clique-Helly circular-arc graphs. The intersection graph of a family of chords in a circle is called a circle graph. Analogously, this class of graphs admits some subclasses too: proper circular-arc graphs, unit circle graphs, Helly circle graphs and clique-Helly circle graphs. In this paper, all possible intersections of these subclasses are studied. After eliminating trivially empty regions, twenty six regions remain. Two of them are empty as a consequence of a theorem by Durán and Lin. Twenty three regions are nonempty and we construct a minimal graph in each of them. Our main result is that the twenty-sixth region is empty, namely we prove that if a graph is Helly circle and unit circle, then it is also a Helly circular-arc graph. Finally, we show that all the trees are included in three of these regions and present an efficient algorithm to classify them.

1 Introduction

A graph G is a *circular-arc graph* if there exists a family R of arcs around a circle and a one-to-one correspondence between vertices of G and arcs in R, such that two distinct vertices are adjacent in G if and only if the corresponding arcs intersect in R. Such a family of arcs is called an *arc*

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representation for G. G is a *proper circular-arc* (PCA) graph if there exists an arc representation for G such that no arc is properly included in another. G is a *unit circular-arc* (UCA) graph if there exists an arc representation for G such that all the arcs are the same length. G is a *Helly circular-arc* (HCA) graph if there exists an arc representation for G such that the arcs verify the Helly property. G is a *clique-Helly circular-arc* ($CHCA$) graph if it is a circular-arc graph and its maximal cliques verify the Helly property.

A graph G is a *circle graph* if there exists a family of chords L on a circle and a one-to-one correspondence between vertices of G and chords of L, such that two distinct vertices are adjacent in G if and only if the corresponding chords intersect in L. Such a family of arcs is called a *chord model* for G. G is a *unit circle* (UC) graph if there exists a chord model for G such that all the chords are the same length. G is a *Helly circle* (HC) graph if there exists a chord model for G such that the chords verify the Helly property. G is a *clique-Helly circle* (CHC) graph if it is a circle graph and its maximal cliques verify the Helly property.

There are several relations defined between these subclasses of circular-arc and circle graphs. Some of them are proved in the following theorems:

Theorem 1.1 ([5]) *If G is a unit circular-arc graph, then it is also a proper circular-arc graph. ($UCA \subseteq PCA$)*

Theorem 1.2 ([3]) *If G is a proper circular-arc graph, then it is also a circle graph. ($PCA \subseteq C$)*

Theorem 1.3 ([1]) *If G is a Helly circle graph, then it is also a clique-Helly circle graph. ($HC \subseteq CHC$)*

Theorem 1.4 ([1]) *G is a unit circle graph if and only if G is a unit circular-arc graph. ($UC \equiv UCA$)*

The following theorems are also useful for this work:

Theorem 1.5 ([1]) *If G is a Helly circle graph, then G is a circle graph and contains no induced diamonds.*

Theorem 1.6 ([2]) *Let G be a graph such that $G \in PCA \backslash UCA$. If $G \in CHCA$, then $G \in HCA$.*

A graph G is a *tree* if it is connected and acyclic. A graph G is an *interval graph* if there exists a family R of intervals on the real line and a one-to-one correspondence between vertices of G and intervals in R, such that two distinct vertices are adjacent in G if and only if the corresponding intervals intersect in R.

Theorem 1.7 ([5]) *Let G be a proper circular-arc graph. Then G is a unit circular-arc graph if and only if G contains no $CI(n, k)$ subgraphs, where n, k are relatively prime and $n > 2k$.*

The definition of $CI(n, k)$ graphs can be found in [1, 5]. We do not transcribe it here because it is beyond the scope of this paper.

2 Intersection of classes

In [1], there is a study of all possible intersections of the subclasses of circular-arc graphs on the one hand, and of the subclasses of circle graphs, on the other. Intersection of classes CA, PCA, UCA, HCA and $CHCA$ defines 13 (not trivially empty) regions, one of which is empty by Theorem 1.6: $(CHCA \cap PCA)\backslash(UCA \cup HCA)$. Intersection of classes C, CHC, HC, PCA and UC defines 10 regions, none of them empty. The core of this work is the study of the intersection between all these classes, thus collapsing the two previous studies into a single, more general one.

Intersection of classes CA, PCA, UCA (same as UC), HCA, $CHCA$, C, CHC and HC defines 26 (not trivially empty) regions. Three of them are empty, and for each one of the remaining 23 cases we show a minimal graph. By "minimal" we mean that no vertex can be deleted from the graph without taking the graph out of its region. Figures 1 and 2 show these 26 regions, along with its graphs. The reader can verify the correctness of the examples by inspection.

3 Empty regions

As a consequence of Theorem 1.6, regions $((PCA\backslash UCA) \cap (CHCA\backslash HC)) \setminus HCA$ and $((PCA\backslash UCA) \cap (CHCA \cap HC)) \setminus HCA$ (see Figure 1) are empty. The following theorem proves that region $(UC \cap HC)\backslash HCA$ is also empty.

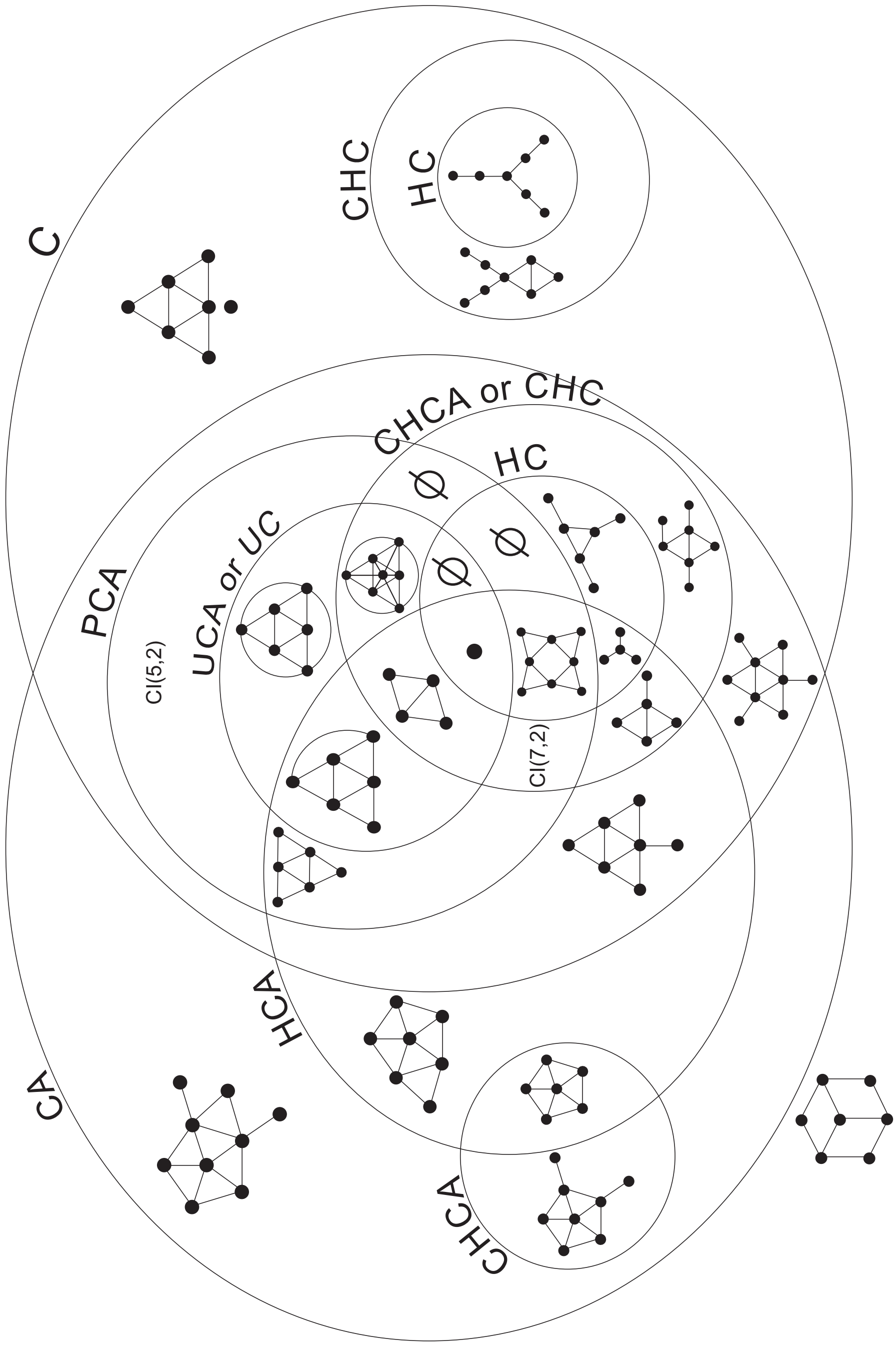


Figure 1: The 26 regions and its minimal graphs. Graphs $CI(5,2)$ and $CI(7,2)$ are shown in Figure 2.

Theorem 3.1 *If G is a unit circle graph and a Helly circle graph, then it is also a Helly circular-arc graph.*

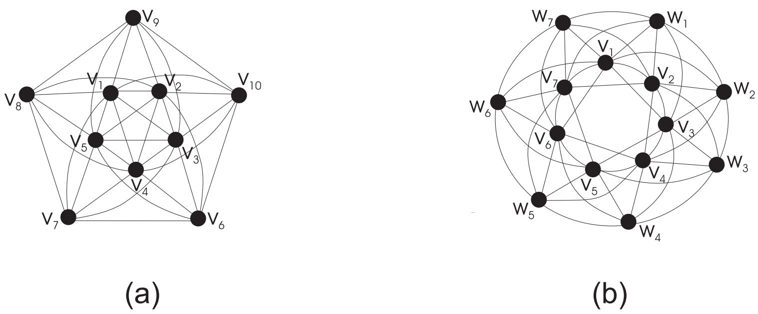


Figure 2: (a) $CI(5,2)$; (b) $CI(7,2)$.

Proof: Throughout this proof, we will use the following conventions: all circles have the same diameter D, L is the length of the sides of the equilateral triangle with vertices laying on the circumference of diameter D (Fig. 3.a), and α_v is the shortest circular arc defined by the ends of chord v (Fig. 3.b).

Let G be a unit circle (UC) and Helly circle (HC) graph, and let R be a UC model for G, with chords of length l. R can be transformed into a unit circular-arc (UCA) representation R', using the procedure shown in the proof of Theorem 1.4: take a UC model for G and transform each chord into the shortest circular arc defined by its ends. The UCA graph corresponding to R' is isomorphic to G. We are going to prove that the arcs in R' verify the Helly property, and as a consequence that $G \in HCA$.

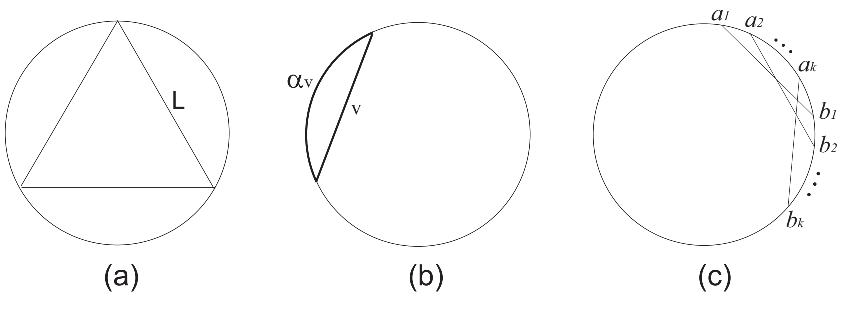


Figure 3: Theorem 3.1.

<u>*Case 1*</u>*:* $l < L$

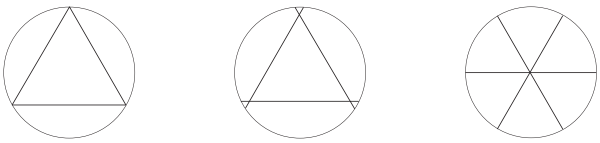
For each maximal clique M in G, order the vertices v_i ($1 \leq i \leq k$, where $k = |M|$) of M, according to the clockwise order of their arcs in R' (Figure

3.c): $\alpha_{v_1} = (a_1, b_1)$, $\alpha_{v_2} = (a_2, b_2)$, ..., $\alpha_{v_k} = (a_k, b_k)$. Fix $a_1 = 0$ as the starting point of the circumference. Since $l < L$, the chords corresponding to the vertices of M cannot complete a turn around the circle. If they did, they would form a cycle of length ≥ 4, and they are supposed to be a clique. Therefore, $a_1 < b_k$. Moreover, since arcs α_{v_i} are all the same length and every chord intersects with all the others: $a_1 < a_2 < ... < a_k < b_1 < b_2 < ... < b_k$. Note that we say $a_k < b_1$, but not $a_k \leq b_1$. This is because we can assume without losing generality that no pair of chords has a common end. In this way, the intersection of all α_{v_i} define an arc, which we call $\gamma_M = (a_k, b_1) = \cap_{1 \leq i \leq k} \alpha_{v_i}$. In other words, for every maximal clique M in R' there exists a non-empty interval γ_M, which implies that R' satisfies the Helly property.

Case 2: $L \leq l \leq D$

2.a: Suppose that G has no induced K_3. In this case, R' trivially satisfies the Helly property.

2.b: Suppose now that G has K_3 as an induced subgraph. Take any of these K_3. In our unit circle model R, this subgraph can be represented in three different ways, as it is shown in Figures 4 (a), (b) and (c) for $l = L$, $L < l < D$ and $l = D$, respectively. We know that $G \in HC$. Starting from the shown models, let us see that the remaining chords must cut every other chord: it cannot cut either none or only one, because $l > L$; and if it cuts more than one, but not all, a diamond would be formed, and then G would not be an HC graph (Theorem 1.5). Therefore, in this case G is complete, and trivially $G \in HCA$. □



(a) (b) (c)

Figure 4: Three chord models for K_3.

4 Trees

In this section, we analyze in which of the studied regions all the trees lay. First of all, we prove that it is possible to find a chord model for every tree.

Theorem 4.1 *If G is a tree, then G is a circle graph.*

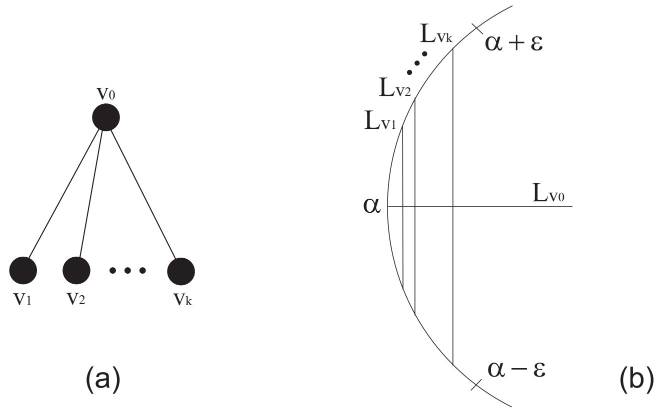


Figure 5: Inductive step in the construction of a chord model.

Proof: This is a constructive proof. Start at the root of the tree, drawing its chord anywhere in the circle. Continue the construction of the chord model in the following way:

Let v_0 be the current node; its chord L_{v_0} is already drawn. If v_0 is a leaf, exit this procedure. Otherwise, let v_1, v_2, ..., v_k be its k children (Figure 5.a). Take one of the ends of L_{v_0}, which we call α. There exists a small enough real number $\varepsilon > 0$, such that the only chord end lying in the circular arc $(\alpha - \varepsilon,\, \alpha + \varepsilon)$ is α. Inside this arc, draw chords L_{v_1}, L_{v_2}, ..., L_{v_k} in parallel, in such a way that every one cuts L_{v_0}, as it is shown in Figure 5.b.

For each node v_1, v_2, ..., v_k, repeat this procedure recursively, until all the nodes in the tree have been drawn. This yields a chord model for every tree. As a consequence, every tree is a circle graph. $\square$

Proposition 4.1 *Trees belong to the regions $UCA \cap HCA \cap HC$, $(HCA \cap HC) \setminus PCA$, or $HC \setminus CA$ (see Figure 6).*

Proof: For the sake of clearness, let us call these three regions R_1, R_2, and R_3, respectively. Figure 1 shows one tree for each one of them. Now let us see that there are no trees in the remaining regions.

We have already seen that every tree is a circle graph. Since trees are acyclic, they do not have K_3 as an induced subgraph, which implies that in every chord model for a tree the chords satisfy trivially the Helly property. As a consequence, every tree is a circular Helly (HC) graph.

Suppose that a tree G is a circular-arc graph. Analogously, since every tree is acyclic, in every arc representation for G the arcs verify the Helly property. Therefore, if a tree is a circular-arc graph, then it is also a Helly circular-arc (HCA) graph.

Theorem 1.7 states that if a graph is PCA but not UCA, then it has $CI(j,k)$ as an induced subgraph, with j,k relatively prime and $j > 2k$. Since all such graphs have induced cycles [5], they are not trees. Therefore, if a tree is PCA, then it is also UCA.

Consequently, trees belong to the regions R_1, R_2, or R_3. □

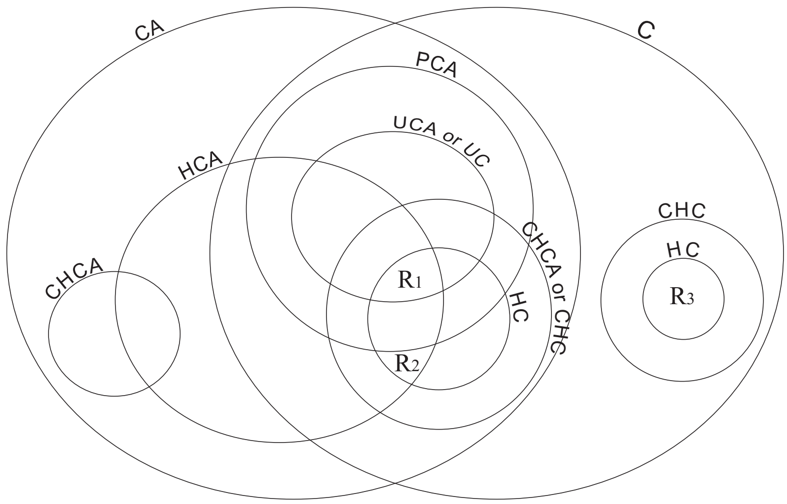


Figure 6: Regions $UCA \cap HCA \cap HC$ ("R_1"), $(HCA \cap HC) \backslash PCA$ ("R_2"), and $HC \backslash CA$ ("R_3").

Proposition 4.2 *A tree belongs to the region $UCA \cap HCA \cap HC$ (i.e., R_1) if and only if it is a path (i.e., every vertex has degree at most 2).*

Proof: It is easy to see that every path belongs to classes UCA, HCA and HC. Now, let G be a tree belonging to these three classes, and suppose that G has at least one vertex v of degree greater than 2. Since G is acyclic, the subgraph induced by v and any three neighbors of v is not a proper circular-arc (PCA) graph, which contradicts the fact that G is a UCA graph. Then, all the vertices in G have degree at most 2, and so G is a path. □

Lekkerkerker and Boland [4] introduced the following characterization for interval graphs:

Theorem 4.2 ([4]) *A graph is an interval graph if and only if it contains none of the graphs shown in Figure 7 as an induced subgraph .*

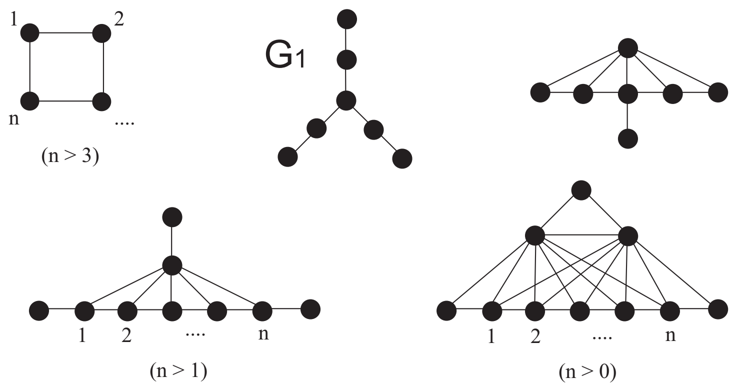


Figure 7: Forbidden structures for interval graphs.

Corollary 4.1 *A tree is an interval graph if and only if it does not contain G_1 (Figure 7) as an induced subgraph.*

Proof: Among the five forbidden structures of Theorem 4.2, only one of them can be an induced subgraph of a tree: G_1. □

Corollary 4.2 *A tree is a circular-arc graph if and only if it is an interval graph.*

Proof: Let G be a circular-arc graph which is a tree and suppose that it is not an interval graph. Therefore, by Theorem 4.2, G contains some of the graphs shown in Figure 7. Since G is a tree, this induced subgraph can only be G_1. But this graph is not a circular-arc graph, which is a contradiction. The converse is true because interval graphs are a subclass of circular-arc graphs. □

Proposition 4.3 *Let G be a tree not belonging to the region $UCA \cap HCA \cap HC$ (i.e., R_1). Then G belongs to the region $(HCA \cap HC) \setminus PCA$ (i.e., R_2) if and only if it does not contain G_1 (Figure 7) as an induced subgraph.*

Proof: Corollaries 4.1 and 4.2 say that a tree is a circular-arc graph if and only if it contains no G_1 as an induced subgraph. Proposition 4.1 says that trees belong to the regions $HC \backslash CA$, $(HCA \cap HC) \backslash PCA$, or $UCA \cap HCA \cap HC$. Let G be a tree not in region $UCA \cap HCA \cap HC$. If G contains G_1, then it is not a circular-arc graph and lays in region $HC \backslash CA$. Otherwise, it is a circular-arc graph and lays in region $(HCA \cap HC) \backslash PCA$. □

Propositions 4.2 and 4.3 give us a method to determine efficiently in which of these three regions a tree G lays. If every vertex in G has degree ≤ 2, it belongs to the region R_1; otherwise, if G contains no G_1 as an induced subgraph, then it belongs to the region R_2; otherwise, G belongs to the region R_3.

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