# New advances about a conjecture on Helly circle graphs

Juan Manuel Barrionuevo [a,1], Aureliano Calvo [a,1] ,
Guillermo Durán [b,2] and Fábio Protti [c,3]

[a] *Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina.*

[b] *Departamento de Ingeniería Industrial, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile.*

[c] *IM and NCE, Universidade Federal do Rio de Janeiro , Rio de Janeiro, Brasil.*

**Abstract**

A circle graph is the intersection graph of a set of chords on a circle. A graph is Helly circle if there is a model of chords satisfying the Helly property. In 2003 it was conjectured that Helly circle graphs are exactly diamond-free circle graphs. In this work, we propose a method to compress any Helly circle representation of a graph and give some ideas to advance in an inductive proof of the diamond conjecture.

*Key words:* circle graphs, compression, diamond, Helly circle graphs

## 1 Introduction

A graph $G$ is called a circle graph ($CG$) if there exists a set of chords $L$ (a model) on a circle and a one-to-one correspondence between vertices of $G$ and

*Email addresses:* `jbarrio@dc.uba.ar` (Juan Manuel Barrionuevo), `ecalvo@dc.uba.ar` (Aureliano Calvo), `gduran@dii.uchile.cl` (Guillermo Durán), `fabiop@nce.ufrj.br` (Fábio Protti).

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chords of $L$ such that two distinct vertices are adjacent if and only if their corresponding chords intersect in the interior of the circle. That is, a circle graph is the intersection graph of a set of chords on a circle.

Circle graphs were introduced in [4], where an application to solve a problem about queues and stacks proposed by Knuth [7] is shown. Polynomial time algorithms for recognizing graphs in this class appear in [1,5,9,10].

A family of subsets S satisfies the Helly property when every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is diamond-free if it does not contain a diamond (a graph formed by four vertices and five edges) as an induced subgraph.

A graph $G$ is a Helly circle ($HC$) graph if there is a model $L$ for $G$ such that the set of chords of $L$ satisfies the Helly property.

A conjecture states that Helly circle graphs are exactly diamond-free circle graphs [2]. Clearly, if a circle graph is $HC$ then it is diamond-free. The converse remains as an interesting open problem. It has been proved that the conjecture holds for permutation graphs, a subclass of circle graphs [8].

Many unsuccessful efforts have been done in order to validate this conjecture for the general case. For example, in [3], the conjecture was restated in terms of an equivalence between two systems of equations and inequations.

In this work, we propose a method to compress any Helly circle representation of a graph and give some ideas to advance in an inductive proof of the conjecture.

## 2 The compression

**Definition 1** *Let $R$ be a representation of a Helly circle graph. $R$ is said to be compressible if and only if there exists another representation $R'$ with the same circular order of the extreme points of the chords such that any given pair of consecutive extreme points of chords can be separated by an arbitrary angle. A $HC$ graph is compressible if and only if all $HC$-representations of the graph are compressible.*

**Definition 2** *Let $R$ be an $HC$-representation. The graph $K_{ja}(R)$ (a subgraph of the clique graph of the original graph) is defined as:*

- *its vertices are the intersection points of the chords in $R$.*
- *two vertices $u$ and $v$ are adjacent if and only if there exists a chord segment in $R$ with extremes $u$ and $v$ such that it does not contain any other*

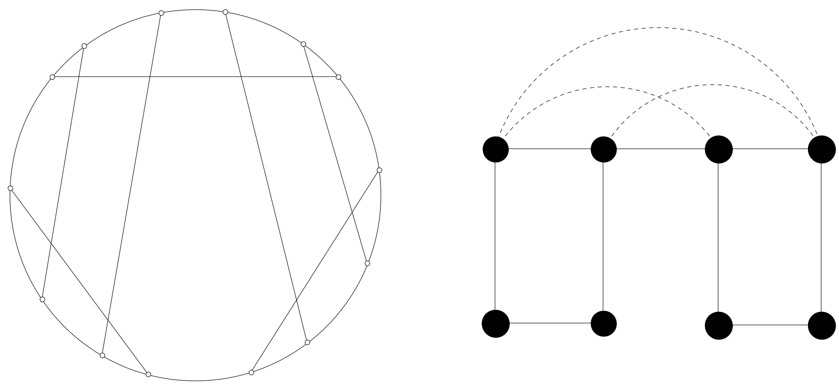

Fig. 1. An $HC$-representation $R$ of a graph $G$ and the $K_{ja}(R)$ graph. Dotted edges are in the clique graph $K(G)$ but not in $K_{ja}(R)$.

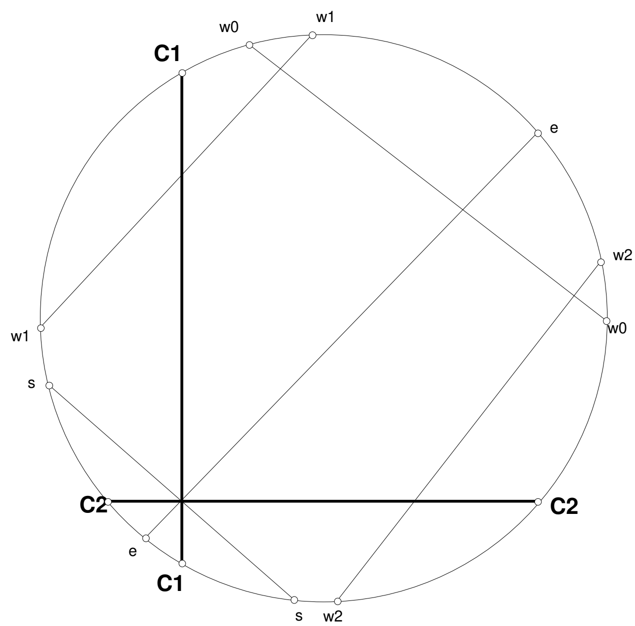


Fig. 2. A reducible path $c1, c2$

*intersection point of the chords.*

**Definition 3** *Given a HC-representation $R$ of a graph $G$, a reducible path $p$ is a path in $G$ such that:*

*(1) every intersection point of chords in $R$ is contained in only one area defined by all the chords in $p$.*
*(2) every clique with its intersection point lying on a chord of the path has one and only one non-simplicial vertex that is not contained in $p$.*
*(3) there exists a simple cycle in $K_{ja}(R)$ that contains every intersection point lying on the chords of $p$*

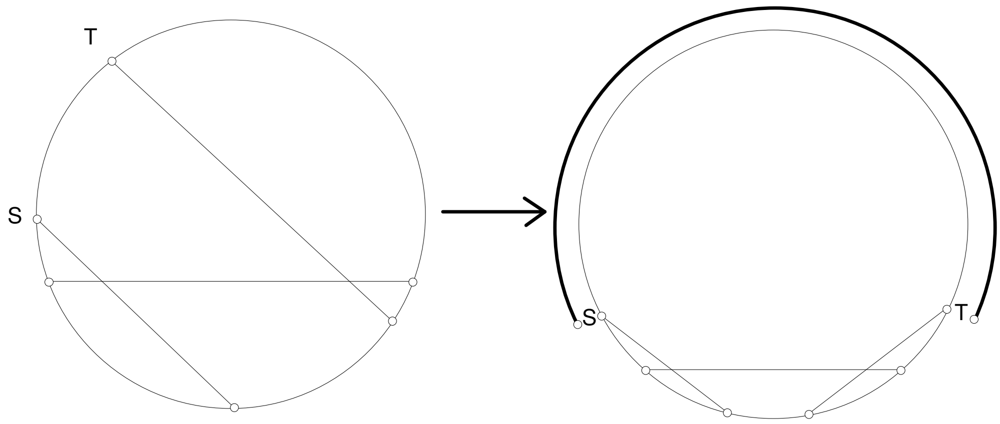


Fig. 3. $P_3$ graph compression

### *2.1 Compression method*

Let us compress the $HC$-representation of a graph $G$. Let $s$ and $t$ be the extremes to be separated by an angle $\alpha$. In order to achieve that, we propose a recursive method.

The base case ($K_1$) is trivial. For the recursive step, we propose different cases:

(1) $G$ is not connected.
Compress each connected component separately.

(2) $\exists v \in G$ such that $v$ only belongs to cliques with two vertices.
$G - \{v\}$ is compressed recursively, and then $v$ is added to the representation of $G - \{v\}$ respecting the circular order of the extreme points of the chords. It should be noted that if $v$ contains either $s$ or $t$, then, in the recursive step, the angle $\alpha$ must be increased with an arbitrarily small $\epsilon$ in order to be able to add $v$ again.

(3) $\exists v \in G$ whose extreme points are not $s$ or $t$, is simplicial, the only clique $c$ that it belongs to is simplicial (as a vertex) in the clique graph $K(G)$, and $|c| \geq 3$.
$G-\{v\}$ is compressed recursively. Then $v$ is added to the new representation placing it infinitely close to another simplicial vertex $v' \in c$. The Helly property is preserved by forcing $v$ to pass through the intersection point defined by $c$.

(4) There exists a reducible path $p$ such that every extreme point of the chords in $p$ or the simplicial nodes in the cliques formed by nodes in $p$ are not $s$ or $t$.
Let $q$ be a set of vertices formed by the vertices whose chords are in $p$ and the simplicial nodes in the cliques lying on the chords of $p$. Compress recursively $G - q$, then add the reducible path $p$ preserving the circular order of the extreme points of the chords and arbitrarily close to each other. Finally, add the simplicial nodes.

### 2.2 A new conjecture

**Conjecture 4** *There are 4 reducible paths on every $HC$-representation of a graph $G$ that has no simplicial vertices.*

Using this conjecture, we prove the compression method's completeness.

**Theorem 5** *If the conjecture 4 is valid, then every $HC$-representation $R$ is compressible using the compression method.*

## 3 The diamond conjeture

As it was said in the introduction, a conjecture states that Helly circle graphs are exactly diamond-free circle graphs [2].

We propose here a way to solve this problem. The property to be proved is that for every circle representation $R$ of a diamond-free circle graph there exists another $HC$-representation $R'$ such that the circular order of the extreme points of the chords is the same as in $R$. We suggest a scheme for an inductive proof:

(1) Prove the Conjecture 4, thus proving the compression method's completeness.
(2) Redefine the concept of reducible path, extending it to all the circle representations of a diamond-free circle graph.
(3) Prove that on every circle representation of a diamond-free graph without simplicial nodes there is a reducible path.
(4) Using the same cases shown on Section 2.1, write an inductive proof. Use the compression to operate with the graphs resulting of the inductive hypothesis.

Using the above scheme, we believe that it is feasible to prove this property, that trivially implies the diamond conjecture [2].

L. Groisman [6] has recently informed us that the compression can be proved using only geometric arguments. In that case, steps 2-3-4 of our proof scheme should be followed in order to prove diamond conjecture.

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