On some subclasses of circular-arc graphs

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Abstract: The intersection graph of a family of arcs on a circle is called a circular-arc graph. This class of graphs admits some interesting subclasses: proper circular-arc graphs, unit circular-arc graphs, Helly circular-arc graphs and clique-Helly circular-arc graphs. In this paper, all possible intersections of these subclasses are studied. There are thirteen regions. Twelve of these are nonempty, and we construct a minimal graph in each of them. Our main result is that the thirteenth region is empty, namely we prove that among proper but no unit circular-arc graphs, every clique-Helly circular-arc graph is also a Helly circular-arc graph.

Keywords: circular-arc graphs, combinatorial problems, graph theory

1- Introduction

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. Intersection graphs have received much attention in the study of algorithmic graph theory and their applications [3]. Well-known special classes of intersection graphs include interval graphs, chordal graphs, circular-arc graphs, permutation graphs, circle graphs, and so on. A circular-arc (CA) graph is the intersection graph of a family of arcs on a circle. We say that these arcs are a circular-arc representation of the graph. We may suppose that they are open. We shall denote a graph G by a pair (V(G),E(G)), where V(G) denotes the set of vertices of G and E(G) the set of edges of G. Let n = |V(G)| and m = |E(G)|. The neighborhood of a vertex v is the set N(v) consisting of all the vertices which are adjacent to v. The closed neighborhood of v is N[v] = N(v) $\cup$ {v}. A clique in a graph G is a set of vertices which induces a maximal complete subgraph of G.

Circular-arc graphs have applications in genetic research [8], traffic control [9], compiler design [14] and statistics [6]. This class of graphs admits some interesting subclasses:

(1) Proper circular-arc graphs: a graph G is a proper circular-arc (PCA) graph if there is a circular-arc representation of G such that no arc is properly contained in any other.

(2) Unit circular-arc graphs: a graph G is a unit circular-arc (UCA) graph if there is a

circular-arc representation of G such that all arcs are of the same length. Clearly, it can be easily proved that UCA $\subseteq$ PCA. In [13], the author showed that this inclusion is strict. An example of a PCA graph which is not a UCA graph has also been given by Golumbic [3].

(3) Helly circular-arc graphs: first, we define the Helly property. A family of subsets S satisfies the Helly property when every subfamily of it consisting of pairwise intersecting subsets has a common element. So, a graph G is a Helly circular-arc (HCA) graph if there is a circular-arc representation of G such that the arcs satisfy the Helly property.

(4) Clique-Helly circular-arc graphs: a graph G is a clique-Helly circular-arc (CH-CA) graph if G is a circular-arc graph and a clique-Helly graph. A graph is clique-Helly when its cliques satisfy the Helly property.

The proposal of this paper is to analize the existence of graphs that belong to exactly k of these four subclasses (k = 0,1,2,3,4). The region R is defined as being any combination of the four subclasses. R is empty if there are not graphs that belong to the k subclasses defined by R and that do not belong to the other 4 - k subclasses. If k = 0, two regions are defined: one of them lying in CA and the other one, out of CA. Then, it has been defined seventeen regions. Four of them are trivially empty because UCA is a subclass of PCA. So, we have thirteen regions that could be non empty. First, we prove the following surprising result: the region defined by the subclasses CH-CA and PCA is empty. Furthermore, the paper shows the existence of minimal members in the other twelve regions (see Figure 8). The minimality of the examples implies that any proper induced subgraph of them belongs to some other region.

This paper is organized in the following way. In Section 2, some theorems that we need in the next section are reviewed. In Section 3, we prove that the region defined by the subclasses CH-CA and PCA is empty and minimal members belonging to the other regions are shown.

Definitions not given here can be found in [3].

2- Preliminaries

In order to characterize the PCA \ UCA region, we need a definition due to Tucker [13]. Let CI(j,k) (j > k) be a circular-arc graph whose representation in circular arcs is built in the following way: let ε be a small positive real number and r=1 the radius of the circle. Draw j arcs $(A_0,A_1,...,A_{j-1})$ of length $l_1=2\pi k/j + \varepsilon$ such that each arc A_i begins in $2\pi i/j$ and finishes in $2\pi(i+k)/j + \varepsilon$ ($A_i = (2\pi i/j,\ 2\pi(i+k)/j + \varepsilon)$) . Then, draw j new arcs $(B_0,B_1,...,B_{j-1})$ of length $l_2=2\pi k/j - \varepsilon$, such that each arc B_i begins in $(2\pi i + \pi k)/j$ and finishes in $(2\pi(i+k)+\pi k)/j - \varepsilon$ ($B_i = ((2\pi i + \pi k)/j,\ (2\pi(i+k) + \pi k)/j - \varepsilon)$). For example, the representation of Figure 1 generates CI(4,1) (Figure 2).

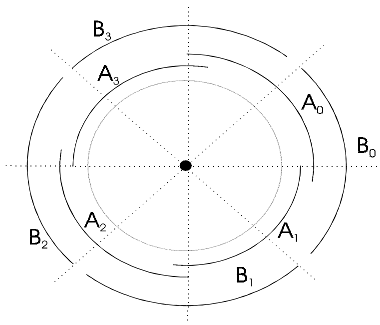

Figure 1

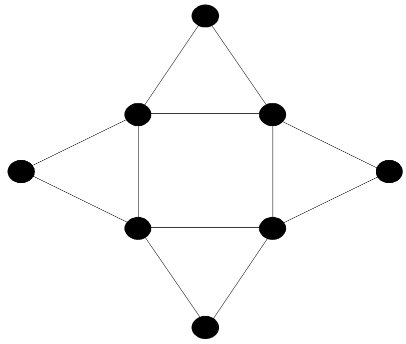

Figure 2

By construction, these graphs are proper circular-arc graphs [13].

Theorem 1 [13]: Let G be a proper circular-arc graph. Then G is a unit circular-arc graph if and only if G contains no CI(j,k) as an induced subgraph, where j,k are relatively prime and $j > 2k$.

The following is a characterization of connected proper circular-arc graphs by local tournaments [1,4,5] and round orientations [1].

A tournament is an orientation of a complete graph. A local tournament is a directed graph in which the out-set as well as the in-set of every vertex is a tournament.

A round enumeration of a directed graph D is a circular ordering $S = \{v_0,...,v_{n-1}\}$ of its vertices such that for each i there exist non-negative integers r_i,s_i such that the vertex v_i has inset $N^S_{in} = \{v_{i-1},v_{i-2},...,v_{i-r_i}\}$ and outset $N^S_{out} = \{v_{i+1},v_{i+2},...,v_{i+s_i}\}$ (additions are subtractions are modulo n). A directed graph which admits a round enumeration is called round. An undirected graph is said to have a round orientation if it admits an orientation which is a round directed graph.

Theorem 2 [1,7]: The following statements are equivalent for a connected graph G

(1) G is orientable as a local tournament.
(2) G has a round orientation.
(3) G is a proper circular arc graph.

We now review a characterization of Helly circular-arc graphs [3]. A matrix has a circular 1's form if the 1's in each column appear in a circular consecutive order. A matrix has the circular 1's property if by a permutation of the rows it can be transformed into a matrix with a circular 1's form. Let G be a graph and $C_1,C_2,...,C_k$ the cliques of G. We will denote by $\mu(C_1,...,C_k)$ the k x n clique matrix, that is the entry (i,j) is 1 if the vertex $v_j \in C_i$ and 0, otherwise.

Theorem 3 [2]: A graph G is a Helly circular-arc graph if and only if $\mu(C_1,...,C_k)$ has the circular 1's property.

3- Circular-arc graphs and their subclasses

3.1 - The empty region

Let us see that the region defined by CH-CA and PCA is empty.

Recall that CI(j,k) ($j > k$) is a circular-arc graph whose representation in circular arcs is built as we defined in Section 2. In order to make the construction independent of ε, we can suppose that the arcs A_i are closed and the arcs B_i are open. We say that there is a 1-1 correspondence between the arc A_i with the vertex a_i and the arc B_i with the vertex b_i of G. So, a_i (i=0,...,j-1) is adjacent to $a_{i-k},b_{i-k},...,a_{i-1},b_{i-1}, b_i,a_{i+1},b_{i+1},...,b_{i+k-1},a_{i+k}$ and b_i is adjacent to $a_{i-k+1},b_{i-k+1},...,a_{i-1},b_{i-1},a_i,a_{i+1},b_{i+1},...,b_{i+k-1},a_{i+k}$ (additions and subtractions have to be understood modulo j).

The following lemma proves that the graph CI(j,k) (j,k relatively prime and $j > 2k$) has only two possible round orientations.

Lemma 1: Let H = CI(j,k) be the graph defined in Theorem 1, with j,k relatively prime and $j > 2k$. Then, H has only two possible round orientations, each one the reverse of the other.

Proof: We say that v dominates w if $N[w] \subseteq N[v]$. In our case, we can see easily that the only vertices dominated by a_i are b_i and b_{i-1} (and they are strictly dominated).

We will prove that there is only one possible circular ordering $S = \{a_0,b_0,a_1,b_1,..., a_{j-1},b_{j-1}\}$ and hence, two possible orientations: either a_i has inset $N^S_{in}(a_i) = \{a_{i-k},b_{i-k},..., a_{i-1},b_{i-1}\}$ and outset $N^S_{out}(a_i) = \{b_i,a_{i+1},...,b_{i+k-1},a_{i+k}\}$, and b_i has inset $N^S_{in}(b_i) = \{a_{i-k+1},b_{i-k+1},...,a_{i-1},b_{i-1},a_i\}$ and outset $N^S_{out}(b_i) = \{a_{i+1},b_{i+1},a_{i+2},...,b_{i+k-1},a_{i+k}\}$; or the reverse orientation.

Let R be a possible circular ordering, $R = \{v_0,v_1,,...,v_{2j-2},v_{2j-1}\}$. We want to see that R = S. If $v_p = a_i$ for any p and i, it will be enough to prove that $\{v_{p-1},v_{p+1}\} = \{b_{i-1},b_i\}$ so that R = S. Recall that $N^R_{out}(v_p) = \{v_{p+1},v_{p+2},...,v_{p+s}\}$ and $N^R_{in}(v_p) = \{v_{p-r},...,v_{p-2},v_{p-1}\}$ because it is a round orientation (now, additions and subtractions have to be understood modulo 2j). Suppose that $b_i \notin \{v_{p-1},v_{p+1}\}$; but b_i is adjacent to a_i, so either $b_i \in \{v_{p+2},...,v_{p+s}\}$ or $b_i \in \{v_{p-r}, ...,v_{p-2}\}$. We analize both possibilities:

(1) $b_i \in \{v_{p+2},...,v_{p+s}\}$. We divide the proof into two cases:

a) $v_{p+1} = b_{i-1}$: as a_{i+k} is adjacent to a_i but it is not adjacent to b_{i-1}, $a_{i+k} \in \{v_{p-r},...,v_{p-1}\}$. But a_{i+k} is adjacent to b_i, so either $a_{i+k} \in N^R_{out}(b_i)$ (in that case, a_i would be an universal vertex because a_i dominates b_i) or $a_{i+k} \in N^R_{in}(b_i)$ (then, a_{i+k} is adjacent to b_{i-1}). In both cases, a contradiction arises.

b) $v_{p+1} \neq b_{i-1}$: v_{p+1} is not dominated by a_i because a_i only dominates b_i and b_{i-1}. Then v_{p+1} has an adjacent vertex w which is adjacent neither to a_i nor to b_i. But, if $w \in N^R_{in}(v_{p+1})$ then w is adjacent to a_i, and if $w \in N^R_{out}(v_{p+1})$ then w is adjacent to b_i. Again, in both cases, a contradiction arises.

(2) $b_i \in \{v_{p-r},...,v_{p-2}\}$. It is an analogous proof to the case (1).

So, $b_i \in \{v_{p-1},v_{p+1}\}$. The proof of $b_{i-1} \in \{v_{p-1},v_{p+1}\}$ is similar.

Now, we can prove the following theorem which asserts that the region defined by CH-CA and PCA is empty

Theorem 4: Let G be a graph $\in$ PCA \ UCA. If G $\in$ CH-CA, then G $\in$ HCA.

Proof: By Theorem 1, G contains CI(j,k) as an induced subgraph, where j,k are relatively prime and j > 2k.. We analyze two different cases. In i), we suppose that $2k < j \leq 3k$ and prove that G is not clique-Helly. In ii), we analize the other case (j > 3k) and prove that G is a Helly circular-arc graph.

i) Let G be a PCA graph such that G contains H = CI(j,k) as an induced subgraph with $2k < j \leq 3k$. We will show that the graph G is not clique-Helly.
a) Let G be isomorphic to H. Clearly, $Ca_i = \{a_i,...,a_{i+k}\} \cup \{b_i,...,b_{i+k-1}\}$ is a clique of G. Let us see that the cliques Ca_0,Ca_k and Ca_{2k} do not verify the Helly property:
$Ca_0 \cap Ca_k \cap Ca_{2k} = \emptyset$ (because $j > 2k$ and $k \geq 1$),
$Ca_0 \cap Ca_k = \{a_k\} \neq \emptyset$,
$Ca_k \cap Ca_{2k} = \{a_{2k}\} \neq \emptyset$ $(j > 2k)$,
$Ca_{2k} \cap Ca_0 \neq \emptyset$ ($a_0 \in Ca_{2k} \cap Ca_0$ because $j \leq 3k$).
b) Let G be not isomorphic to H. As H is a connected PCA graph, it has a round orientation (Theorem 2). By Lemma 1, there are only two possible round orientations, each one the reverse of the other. We use one of them (Figure 3)

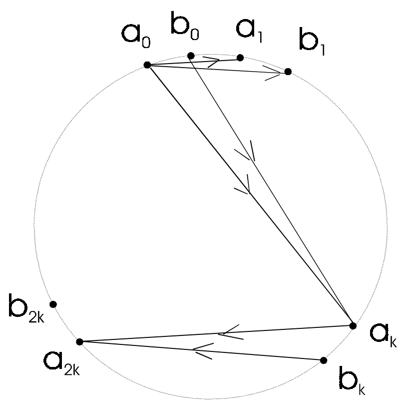

Figure 3

By a), H contains the cliques Ca_0,Ca_k and Ca_{2k} and they do not verify the Helly property. We extend the complete subgraphs Ca_0,Ca_k and Ca_{2k} to cliques in G (and call them $C'a_0$,$C'a_k$ and $C'a_{2k}$). Suppose that G is clique-Helly, then there is a vertex $v \in V(G) \setminus V(H)$ such that $v \in C'a_0 \cap C'a_k \cap C'a_{2k}$. As Ca_0, Ca_k and Ca_{2k} cover all the vertices in H, the vertex v is adjacent to w, for any $w \in V(H)$. If we add the vertex v to H, the new graph is a connected PCA too. Then, by Theorem 2, there is a round orientation of the subgraph induced by the vertices of H and v (we use the round orientation of the Figure 3 and add the vertex v). We can suppose, without loss of generality, that v is added between a_0 and b_o

(Figure 4). By inspection, we can see that vertex v is not adjacent to b_k, otherwise it would not be a round orientation. So, a contradiction arises because v is adjacent to w, for any w $\in$ V(H).

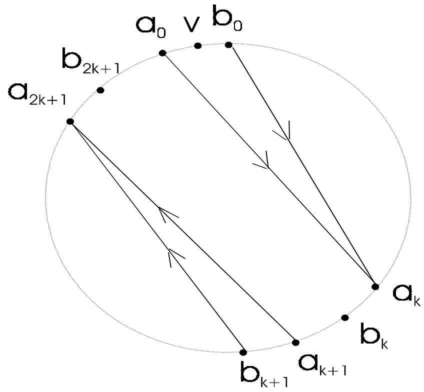

Figure 4

ii) Let G be a PCA graph such that G contains H = CI(j,k) as an induced subgraph with 3k < j. We will prove that G is a HCA graph. Suppose that G is not a Helly circular-arc graph. Then, any circular-arc representation does not verify the Helly property. Let $v_1,...,v_t$ be a minimal subset of vertices of G whose corresponding arcs are not Helly in a given representation. Each v_i may belong or not to H. We analyze two cases:

a) t = 3: the arcs A'_1, A'_2 and A'_3 (correspoding to the vertices v_1,v_2 and v_3) minimally non Helly are depicted in Figure 5.

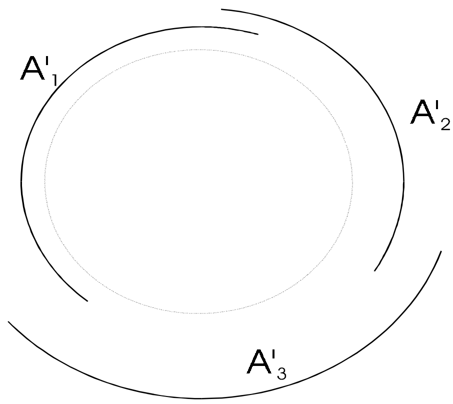

Figure 5

Then v_1,v_2 and v_3 induce a triangle and every vertex of G is adjacent to either v_1 or v_2 or v_3. As G is a connected PCA graph, the subgraph G' induced by the vertices of H and v_1,v_2 and v_3 is a connected PCA graph too. Then, by Theorem 2, there is a round orientation of G'. We use the round orientation of Figure 4 (one of the two possible round orientations) and add v_1, v_2 and v_3 if they are not in H. As j > 3k and v_1,v_2,v_3 form a triangle, these three vertices must be located in the round orientation between a_i and a_{i+k}. We may suppose, without loss of generality, that i = 0 (see Figure 6).

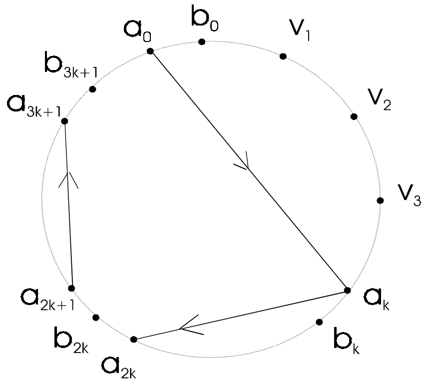

Figure 6

Hence, there is a vertex b_t (b_{2k}, in this case) which is adjacent neither to v_1, v_2 nor v_3, contradicting the above assertion.

b) t $\geq$ 4: as these arcs are minimally non Helly, there are two of them (A'_1 and A'_2) which cover all the circle (Figure 7). Hence, every vertex of G is adjacent to either v_1 or v_2, with v_1 adjacent to v_2. So, the same contradiction of the case a) arises.

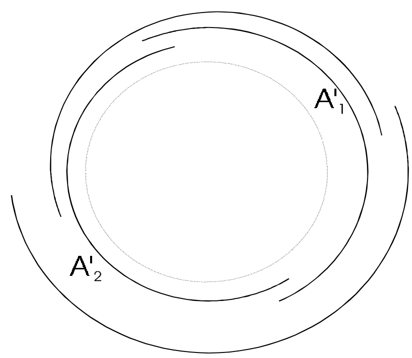

Figure 7

Corollary: The region defined by CH-CA and PCA is empty

Proof: It is a trivial consequence of Theorem 4.

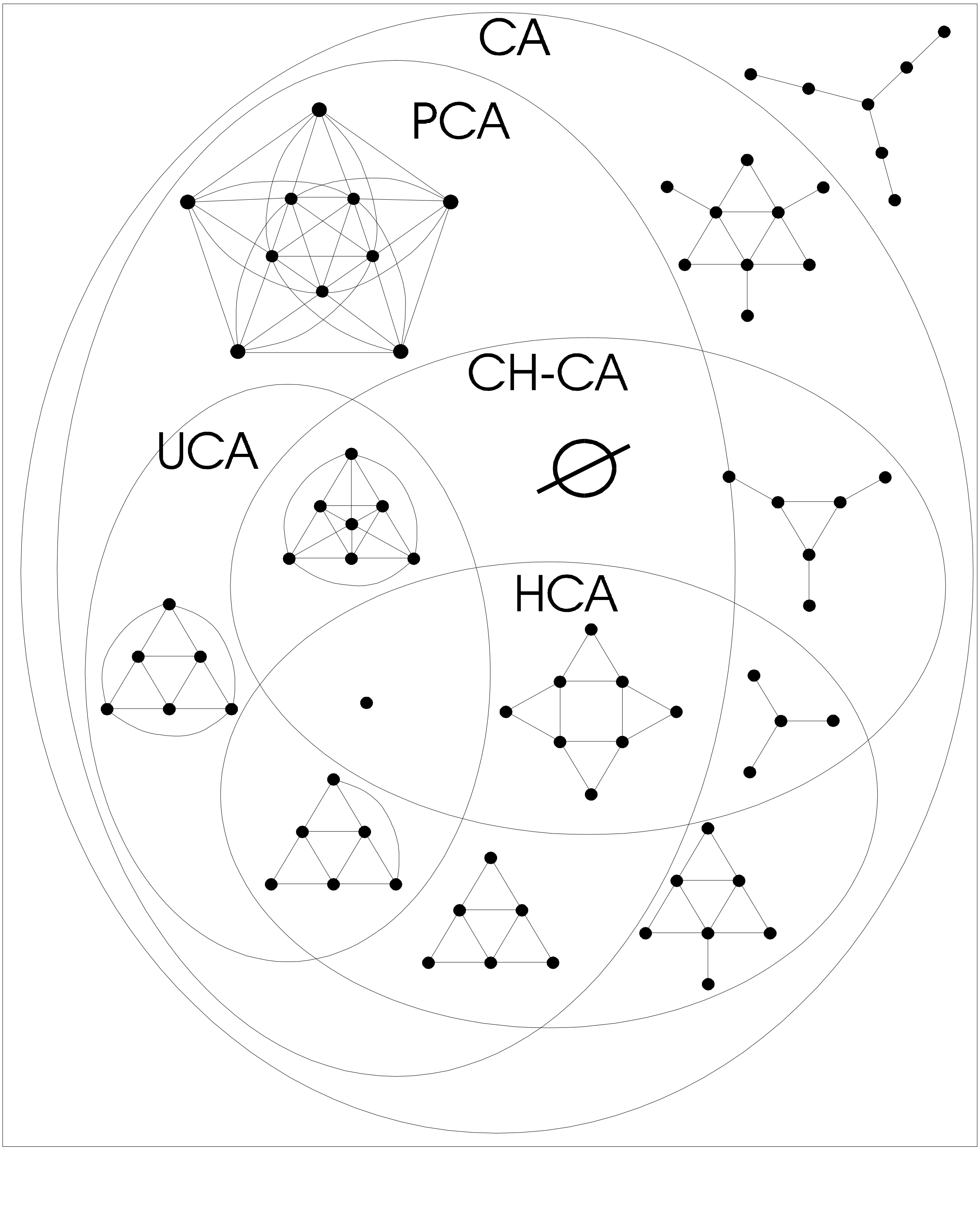

Figure 8: A minimal member for each region

3.2 - Minimal examples for each region

The proofs that the graphs of Figure 8 belongs to the corresponding region and the minimality of these examples can be verified easily by the reader using characterization theorems of circular-arc graphs and their subclasses [2,10,11,12,13]. We present here only two proofs of minimal members belonging to the respective regions.

Proposition 1: Graph H_1 (Figure 9) belongs to the region defined by PCA.

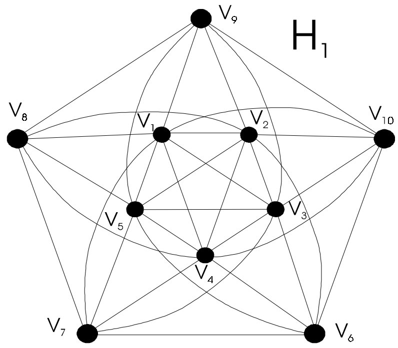

Figure 9: H_1 is Tucker's CI(5,2) graph

Proof:

(1) H_1 is a proper circular-arc graph but it is not a unit circular-arc graph: by [13], CI(5,2) $\in$ PCA \ UCA.

(2) H_1 is not a clique-Helly circular-arc graph: the subfamily of cliques $C_1=\{v_2,v_3,v_4,v_5,v_6\}$, $C_2=\{v_1,v_3,v_4,v_5,v_7\}$, $C_3=\{v_1,v_2,v_4,v_5,v_8\}$, $C_4=\{v_1,v_2,v_3,v_5,v_9\}$, $C_5=\{v_1,v_2,v_3,v_4,v_{10}\}$ does not verify the Helly property.

(3) H_1 is not a Helly circular-arc graph: suppose the contrary. In order to draw a circular-arc representation of the induced cycle C formed by vertices v_6,v_7,v_8,v_9,v_{10}, we need to cover all the circle (Figure 10). Furthermore, the arcs representing the vertices of each clique must have a common intersection because the graph is a Helly circular-arc graph. Both properties imply that each clique contains any vertex of the induced cycle C. Then, the clique K_5 induced by vertices $\{v_1,v_2,v_3,v_4,v_5\}$ leads us to a contradiction.

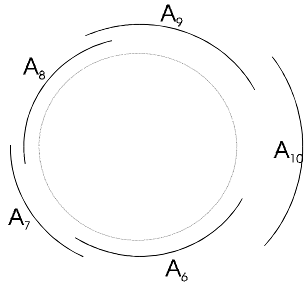

Figure 10

In order to verify the minimality of this example, we are going to prove that any proper induced subgraph H of H_1 is UCA. So, let us see that H does not contain CI(j,k) as an induced subgraph, where j,k are relatively prime and $j > 2k$. It is enough to prove this fact for CI(4,1) and CI(3,1) because they are the only graphs of this family with at most nine vertices. But H has a maximum independent set of size at most two and these two graphs have maximum independent set of cardinality 4 and 3, respectively.

Proposition 2: Graph H_2 (Figure 11) belongs to the region defined by CH-CA, PCA and UCA.

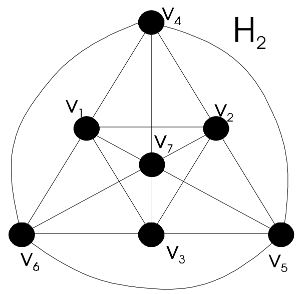

Figure 11

Proof:

(1) H_2 is a unit circular-arc graph: let r=1 be the radius of the circle and $l=3/4\ \pi$ the common length of each arc, corresponding to each vertex of the graph. Figure 12 shows a unit circular-arc representation of it.

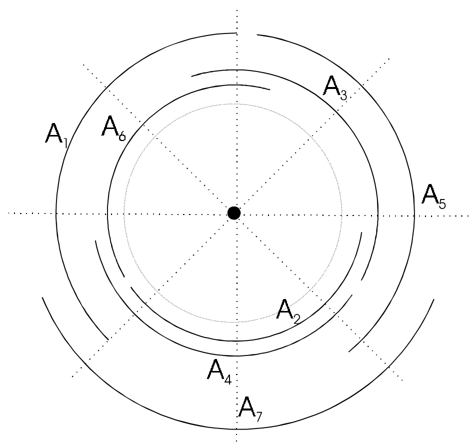

Figure 12

(2) H_2 is a clique-Helly circular-arc graph: by (1), H_2 is a circular-arc graph. We want to prove that it is a clique-Helly graph. But H_2 contains an universal vertex (v_7), then we can assert that it is a clique-Helly graph.

(3) H_2 is not a Helly circular-arc graph: let us see that the subgraph induced by the set of vertices $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ is not a Helly circular-arc graph. Suppose the contrary; as v_1, v_2, v_5 and v_6 induce a cycle C_4, the respective arcs in the circular-arc representation must be drawn as in the Figure 13. Furthermore, the vertex v_3 is adjacent to all of them. So, in a Helly representation, A_3 (arc corresponding to v_3) must intersect $A_1 \cap A_2$, $A_2 \cap A_5$, $A_5 \cap A_6$ and $A_6 \cap A_1$. This fact implies that A_3 contains at least one of the other arcs. Without loss of generality, it can be supposed that this arc is A_5 (Figure 13). Now, it is not possible that A_4 (arc corresponding to v_4) is added to the Helly circular-arc representation such that it has non-empty intersection with A_5 and an empty intersection with A_3.

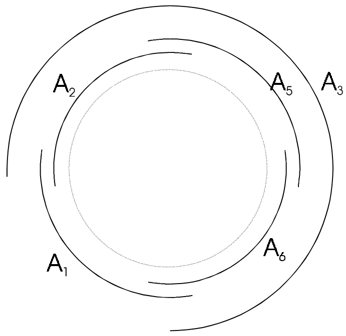

Figure 13

So, H_2 is not a Helly circular-arc graph because this is an hereditary property..

Let us verify the minimality of H_2. If a proper induced subgraph H does not contain either v_1, v_2, v_3, v_4, v_5 or v_6, a circular 1's clique matrix can be easily found, so H is a HCA graph (Theorem 3). On the other hand, the only proper induced subgraph that contains all these vertices is the subgraph induced by the set of vertices $\{v_1,v_2,v_3,v_4,v_5,v_6\}$, which belongs to another region (see Figure 8).

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