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On Balanced Graphs

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Abstract. Berge defined a hypergraph to be balanced if its incidence matrix is balanced. We consider this concept applied to graphs, and call a graph to be balanced when its clique matrix is balanced. Characterizations of balanced graphs by forbidden subgraphs and by clique subgraphs are proved in this work. Using properties of domination we define four subclasses of balanced graphs. Two of them are characterized by 0-1 matrices and can be recognized in polynomial time. Furthermore, we propose polynomial time combinatorial algorithms for the problems of stable set, clique-independent set and clique-transversal for one of these subclasses of balanced graphs. Finally, we analyse the behavior of balanced graphs and these four subclasses under the clique graph operator.

Key words. algorithms, balanced graphs, balanced hypergraphs, clique graphs, domination, 0-1 matrices.

1. Introduction

A clique in a graph is a complete subgraph maximal with respect to inclusion. The clique number of a graph G is the cardinality of a maximum clique of G and is denoted by $\omega(G)$.

The chromatic number of a graph G is the smallest number of colours that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same colour. An obvious lower bound is the clique number of G.

Berge [2] called a graph G perfect whenever the chromatic number of every induced subgraph H of G is equal to $\omega(H)$. Perfect graphs are very interesting from an algorithmic point of view. While determining the clique number and the chromatic number of a graph are NP-complete problems, they are solvable

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in polynomial time for perfect graphs [20]. For more background information on algorithms on perfect graphs see [19].

A sequence $v_1, \ldots, v_k$ of distinct vertices ($k \geq 3$) is a cycle in a graph G if $(v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v_1)$ are edges of G. These edges are called the edges of the cycle. The length of the cycle is the number k of its edges. An odd cycle is a cycle of odd length. In subsequent expressions concerning cycles, all index arithmetic is done modulo the length of the cycle.

A chord of a cycle is an edge between two vertices of the cycle that is not an edge of the cycle. A cycle is chordless if it contains no chords. Chordless cycles of length greater than 3 are called holes.

A sequence $v_1, E_1, \ldots, v_k, E_k$ of distinct vertices $v_1, \ldots, v_k$ and distinct hyperedges $E_1, \ldots, E_k$ of a hypergraph H is a special cycle of length k if $k \geq 3$, $v_i, v_{i+1} \in E_i$ and $E_i \cap \{v_1, \ldots, v_k\} = \{v_i, v_{i+1}\}$, for each i, $1 \leq i \leq k$.

Let A be a 0-1 matrix. We say that the row i is included in the row k if for every column j, $A(i, j) = 1$ implies $A(k, j) = 1$.

Let $M_1, \ldots, M_k$ and $v_1, \ldots, v_n$ be the cliques and vertices of a graph G, respectively. We define A_G, a clique matrix of G, as a 0-1 matrix whose entry (i, j) is 1 if $v_j \in M_i$, and 0 otherwise.

A 0-1 matrix M is balanced if it does not contain the vertex-edge incidence matrix of an odd cycle as a submatrix. A 0-1 matrix M is totally balanced if it does not contain the vertex-edge incidence matrix of a cycle as a submatrix.

In 1969 (c.f. [15]), Berge defined a hypergraph to be balanced if its incidence matrix is balanced, or equivalently, if it contains no special cycles of odd length. See [4,5]. Applying this concept to graphs, one obtains the class of balanced graphs, formed by those graphs having a balanced clique matrix. Note that balanced graphs are well defined, since if the clique matrix of a graph is balanced then all its clique matrices are balanced. Balanced graphs were considered in [14].

The clique hypergraph of a graph G has the same vertex set as G and all cliques of G as hyperedges. Clearly, a graph G is balanced if and only if its clique hypergraph is balanced.

Let v, w be vertices of G. Denote by $M(G)$ the set of cliques of G, by $M(v)$ the set of cliques of G that contain v, and by $M(v, w)$ the set of cliques of G that contain v and w. Vertices that belong to exactly one clique will be called simplicial vertices.

The neighborhood of a vertex v in a graph G is the set $N_G(v)$ consisting of all the vertices that are adjacent to v. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The common neighborhood and the closed common neighborhood of an edge $e = (v, w)$ are $N_G(e) = N_G(v) \cap N_G(w)$ and $N_G[e] = N_G[v] \cap N_G[w]$, respectively, and, in a more general way, the common neighborhood and the closed common neighborhood of a non empty subset of vertices W are $N_G(W) = \bigcap_{w \in W} N_G(w)$ and $N_G[W] = \bigcap_{w \in W} N_G[w]$, respectively. We define $N_G(\emptyset) = N_G[\emptyset] = V(G)$.

A clique-transversal of a graph G is a subset of vertices intersecting all the cliques of G. A clique-independent set is a subset of pairwise disjoint cliques of G. Denote by $\tau_{\mathrm{c}}(G)$ and $\alpha_{\mathrm{c}}(G)$ the cardinalities of the minimum clique-transversal and maximum clique-independent set of G, respectively. A graph G is clique-perfect when $\tau_{\mathrm{c}}(H) = \alpha_{\mathrm{c}}(H)$, for every induced subgraph H of G [21].

A family of subsets satisfies the Helly property when every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is clique-Helly when its cliques satisfy the Helly property. A graph is hereditary clique-Helly (HCH) when H is clique-Helly for every induced subgraph H of G.

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.

The clique graph $K(G)$ of G is the intersection graph of the cliques of G. We can define $K^j(G)$ as the j-th iterated clique graph of G, where $K^1(G) = K(G)$ and $K^j(G) = K(K^{j-1}(G))$, $j \geq 2$.

Let $\mathcal{H}$ be a class of graphs. The notation $K(\mathcal{H})$ means the class of clique graphs of the graphs in $\mathcal{H}$, and $K^{-1}(\mathcal{H})$ the class of graphs whose clique graphs are in $\mathcal{H}$.

Let C be a set of cliques of G. The clique subgraph of a graph G generated by C is the graph formed by the vertices and edges of C. A clique subgraph of G is not necessarily an induced subgraph of G.

A graph G is an interval graph if G is the intersection graph of a finite family of intervals of the real line.

A graph G is trivially perfect if for all induced subgraphs H of G, the cardinality of the maximum stable set of H is equal to the number of cliques of H.

A 0-1 matrix M is totally unimodular if the determinant of each square submatrix of M is 0, 1 or -1.

A graph G is totally unimodular if its clique matrix is totally unimodular.

Since the determinant of the vertex-edge incidence matrix of an odd cycle is ± 2, totally unimodular matrices are balanced matrices and then totally unimodular graphs are balanced graphs.

Interval graphs and trivially perfect graphs are totally unimodular graphs [19] and, therefore, they are balanced graphs.

A graph is bipartite when it contains no cycles of odd length. A graph is strongly chordal when it is chordal and each of its cycles of even length at least 6 has an odd chord [17]. Such a class corresponds exactly to totally balanced graphs, i.e., graphs whose clique matrices are totally balanced [1]. Bipartite graphs and strongly chordal graphs form important subclasses of balanced graphs.

The organization of the paper is the following. In Section 2, we describe background properties of balanced graphs.

In Section 3, new characterizations of balanced graphs are presented. One of them is by forbidden subgraphs and the other is by clique subgraphs.

In Section 4, four subclasses of balanced graphs are introduced using simple properties of domination. We analyse the inclusion relations between them. Two of these classes are characterized using 0-1 matrices and the characterizations lead to polynomial recognition algorithms. In the final part of this section, we present a combinatorial algorithm for the maximum stable set problem for one of these subclasses.

Finally, in the last section, we study the clique graphs of balanced graphs and these four subclasses. As a corollary of these results, we deduce the existence of combinatorial algorithms for the maximum clique-independent set and the minimum clique-transversal problems for one of these subclasses of balanced graphs.

2. Preliminaries

A characterization of hereditary clique-Helly graphs can be formulated in the following way:

Theorem 1. [27] *A graph G is hereditary clique-Helly if and only if A_G does not contain a vertex-edge incidence matrix of a 3-cycle as a submatrix.*

This theorem implies the following result.

Corollary 1. *Let G be a balanced graph. Then G is hereditary clique-Helly.*

In [27], it is also proved that no connected hereditary clique-Helly graph has more cliques than edges. Then this result holds.

Corollary 2. *Let G be a connected balanced graph. Then the number of cliques of G is at most the number of edges of G.*

There exists an algorithm which calculates all the cliques of a graph in $O(mnk)$ time where m is the number of edges, n the number of vertices and k the number of cliques [31] (the algorithm generates each clique sequentially in $O(mn)$ time). So a clique matrix of a hereditary clique-Helly graph can be computed in polynomial time in the size of the graph. On the other hand, Conforti, Cornuéjols, and Rao formulated a polynomial-time recognition algorithm for balanced 0-1 matrices [11]. These two algorithms and the fact that hereditary clique-Helly graphs have no more than m cliques imply the following result.

Corollary 3. [14] *There is a polynomial-time recognition algorithm for balanced graphs.*

It should be mentioned that clique matrices where characterized by P. C. Gilmore in the 60's, c.f. [13].

It is not difficult to see that the clique matrix of a graph G and the clique matrix of an induced subgraph of G are related in the following way:

Lemma 1. *Let G be a graph and H an induced subgraph of G. Then A_H is the submatrix of A_G obtained by keeping the columns corresponding to the vertices of H and removing the included rows.*

On the other hand, for hereditary clique-Helly graphs, the clique matrix of a graph G and the clique matrix of a clique subgraph of G are related in the following way:

Theorem 2. [27] *Let G be a hereditary clique-Helly graph and S a subset of its cliques. Let H be the clique subgraph of G formed by the vertices and edges of S. Then A_H is the submatrix of A_G obtained by taking the rows corresponding to the cliques in S and the columns corresponding to the vertices of these cliques.*

Since a submatrix of a balanced matrix is also balanced, these results imply that balanced graphs are closed under induced subgraphs and clique subgraphs.

Let e_k be a vector with k 1's. A matrix $M \in R^{k \times n}$ is perfect if the polyhedron $P(M) = \{x/x \in R^n, Mx \leq e_k, x \geq 0\}$ has only integer extrema.

Fulkerson, Hoffman and Oppenheim [18] proved the following result which implies that balanced matrices are perfect matrices.

Theorem 3. [18] *If M is a balanced matrix, then the polyhedra $P(M) = \{x/x \in R^n, Mx \leq e_k, x \geq 0\}$ and $Q(M) = \{x/x \in R^n, Mx \geq e_k, x \geq 0\}$ have only integer extrema.*

On the other hand, Chvátal [9] proved the theorem below that connects perfect matrices with perfect graphs.

Theorem 4. [9] *A graph G is perfect if and only if its clique matrix is perfect.*

By Theorems 3 and 4, balanced graphs are perfect graphs.

A 0-1 matrix A is k-colourable if there exists a k-colouring of its columns such that for every row i that has at least two 1s in columns corresponding to colours J and L, there are entries $a_{ij} = a_{il} = 1$, where column j has colour J and column l has colour L.

Berge proved the following theorem.

Theorem 5. [3] *A 0-1 matrix A is balanced if and only if every submatrix of A is k-colourable for every k.*

Based on the proof of Theorem 5 and using the bicolouring algorithm of Cameron and Edmonds [8], a balanced matrix can be efficiently k-coloured [12]. Since it is not difficult to see that for a graph G a $\chi(G)$-colouring of A_G gives an $\chi(G)$-colouring of G, and in a balanced graph G a $\chi(G)$-colouring of G is equivalent to a $\omega(G)$-colouring of G and $\omega(G)$ can be easily calculated, so there exists a polynomial time combinatorial algorithm to find an optimal colouring of a balanced graph [10].

Berge and Las Vergnas proved in [6] a theorem about balanced hypergraphs which can be formulated in terms of graphs in the following way:

Theorem 6. [6] *If G is a balanced graph then $\tau_c(G) = \alpha_c(G)$.*

Corollary 4. *Balanced graphs are clique-perfect.*

Moreover, the clique-transversal number $\tau_c(G)$ (and hence the clique-independence number $\alpha_c(G)$) of a balanced graph G can be determined polynomially by linear programming [14].

3. New characterizations of balanced graphs

In this section, two new characterizations of balanced graphs are presented. The first one, by forbidden subgraphs and the second one, by clique subgraphs.

A sun (or trampoline) is a chordal graph G on $2r$ vertices whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that W is a stable set and for each i and j, w_j is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod{r}$. A sun is odd if r is odd.

Some subclasses of balanced graphs are characterized by forbidden subgraphs. We can see that in the following two theorems.

Theorem 7. [17] *A graph is strongly chordal if and only if it is sun-free chordal.*

Theorem 8. [25] *A graph is chordal and balanced if and only if it is odd sun-free chordal.*

An *extended odd sun* is an odd cycle C and a subset of pairwise adjacent vertices $W_e \subseteq N_G(e) \setminus C$ for each edge e of C, such that $N_G(W_e) \cap N_G(e) \cap C = \emptyset$ and $|W_e| \leq |N_G(e) \cap C|$.

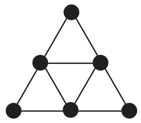

Fig. 1. Hajös graph.

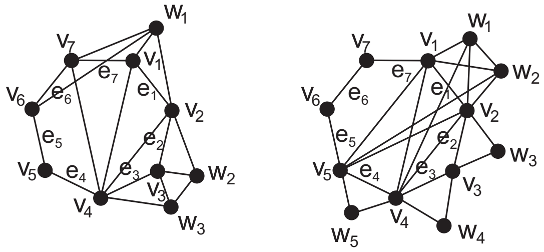

Fig. 2. Two examples of graphs that are not balanced. In the first one, $W_{e_1} = W_{e_7} = \{w_1\}, W_{e_2} = \{w_2\}, W_{e_3} = \{w_3\}$ and $W_{e_4} = W_{e_5} = W_{e_6} = \emptyset$. In the second one, $W_{e_1} = \{w_1, w_2\}, W_{e_2} = \{w_3\}, W_{e_3} = \{w_4\}, W_{e_4} = \{w_5\}$ and $W_{e_5} = W_{e_6} = W_{e_7} = \emptyset$.

Clearly, odd suns are extended odd suns. The smallest extended odd sun is the Hajös graph (Figure 1).

Other examples of extended odd suns appear in Figure 2. Note that the subsets W_e and W_f, corresponding to the edges e and f respectively, may overlap.

Theorem 9. *A graph is balanced if and only if it does not contain an extended odd sun as an induced subgraph.*

Proof. Let G be a graph. Suppose that G has the following extended odd sun: an odd cycle $C = \{v_1, \ldots, v_{2k+1}\}$ and a subset of pairwise adjacent vertices $W_i \subseteq N_G(e_i) \setminus C$ for each edge $e_i = (v_i, v_{i+1})$ of C, such that $N_G(W_i) \cap N_G(e_i) \cap C = \emptyset$.

Let $e_i = (v_i, v_{i+1})$ be an edge of C. Then $\{v_i, v_{i+1}\} \cup W_i$ is contained in a clique M_i of G, and $M_i \cap C = \{v_i, v_{i+1}\}$ because $N(e_i) \cap N(W_i) \cap C = \emptyset$.

Now, if we choose the rows of A_G corresponding to $M_1, \ldots, M_{2k+1}$ and the columns of A_G corresponding to $v_1, \ldots, v_{2k+1}$, we have a vertex-edge incidence matrix of an odd cycle as a submatrix of A_G. So, A_G is not balanced, and thus G is not balanced.

Conversely, suppose that G is not a balanced graph, and then A_G is not a balanced matrix. So, we have the following submatrix A' in A_G, where $M_1, \ldots, M_{2k+1}$ are cliques of G and $v_1, \ldots, v_{2k+1}$ are vertices of G:

Thus $v_1, \ldots, v_{2k+1}$ is an odd cycle C of G and M_i is a clique such that $M_i \cap C = \{v_i, v_{i+1}\}$. Let e_i be the edge (v_i, v_{i+1}). Then either $N_G(e_i) \cap C = \emptyset$ and then we

	v_1	v_2	v_3	$\ldots$	v_{2k+1}
M_1	1	1	0	$\ldots$	0
M_2	0	1	1	$\ldots$	0
M_3	0	0	1	$\ldots$	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
M_{2k+1}	1	0	0	$\ldots$	1

Fig. 3. Vertex-edge incidence matrix of an odd cycle

define W_i to be the empty set, or for each $v \in N_G(e_i) \cap C$ there is a vertex w in M_i non-adjacent to v, and those vertices form a subset of pairwise adjacent vertices $W_i \subseteq N_G(e_i) \setminus C$ such that $N_G(W_i) \cap N_G(e_i) \cap C = \emptyset$ and $|W_i| \leq |N_G(e_i) \cap C|$. □

Remark 1. Extended odd suns are not necessarily minimal. The Hajös graph is an induced subgraph of the extended odd sun of Figure 4.

Theorem 10. *A graph G is balanced if and only if G is hereditary clique-Helly and does not contain a clique subgraph with an odd hole.*

Proof. $\Rightarrow$) Let G be a balanced graph. By Corollary 1, G is HCH. Let H be a clique subgraph of G. Since balancedness is hereditary for clique subgraphs, H is balanced. Since induced subgraphs of G are also balanced, H can not contain an odd chordless cycle of length ≥ 5 as an induced subgraph.

$\Leftarrow$) Suppose that G is not a balanced graph, thus A_G is not a balanced matrix. If A_G contains the vertex-edge incidence matrix of a 3-cycle as a submatrix, then G is not HCH. Otherwise, G is HCH and A_G contains the vertex-edge incidence matrix of an odd hole as a submatrix A' (Figure 3, with $k \geq 2$). Let H be the clique subgraph of G formed by the cliques of G corresponding to the rows of A', and let H' be the subgraph of H induced by the vertices corresponding to the columns of A' (these vertices are vertices of H by the construction of A'). By Theorem 2, the clique matrix A_H is the submatrix of A_G obtained by keeping the rows of A' and then removing the null columns. Now, by Lemma 1, the clique matrix $A_{H'}$ is A'. Thus H' is an odd hole. □

4. Graph Classes: VE, EE, VV and EV

In this section, we define and study four classes of graphs, based on simple domination properties. These graphs form natural subclasses of balanced graphs.

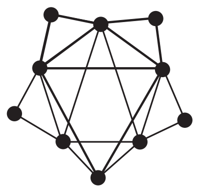

Fig. 4. An extended odd sun which is not minimal.

Let v, w be vertices and e, f edges of a graph G. Say that vertex v (edge e) dominates vertex w (edge f) when $N_G[v] \supseteq N_G[w]$ ($N_G[e] \supseteq N_G[f]$). Similarly, vertex v (edge e) dominates edge f (vertex w) when $N_G[v] \supseteq N_G[f]$ ($N_G[e] \supseteq N_G[w]$).

A graph G is a VE graph if any odd cycle of G contains a vertex that dominates some edge of the cycle, where the edge is non-incident to the vertex.

A graph G is an EV graph if any odd cycle of G contains an edge that dominates some vertex of the cycle.

Finally, a graph G is a VV (EE) graph if any odd cycle of it contains a vertex (edge) that dominates some other vertex (edge) of the cycle.

4.1. Inclusion relations

Let us see the inclusion relations between these graph classes.

Theorem 11. *Let G be an EV graph. Then G is an EE graph and a VV graph.*

Proof. Let $C = \{v_1, \ldots, v_{2j+1}\}$ be an odd cycle of G. By hypothesis, as G is an EV graph, there is an edge $e = (v_i, v_{i+1})$ of C that dominates a vertex v_k of C. Then $e = (v_i, v_{i+1})$ dominates $e_1 = (v_{k-1}, v_k)$ and $e_2 = (v_k, v_{k+1})$, and at least one of these edges is not equal to e. So, G is an EE graph. On the other hand, v_i and v_{i+1} dominate v_k, and at least one of them is different from v_k. In consequence, G is a VV graph too. $\square$

Theorem 12. *Let G be an EE graph. Then G is a VE graph.*

Proof. Let $C = \{v_1, \ldots, v_{2j+1}\}$ be an odd cycle of G. By hypothesis, as G is an EE graph, there is an edge $e = (v_i, v_{i+1})$ that dominates an edge $f = (v_k, v_{k+1})$ of C ($e \neq f$). We may suppose that $v_i \neq v_{k+1}$, so v_i dominates $f = (v_k, v_{k+1})$, which implies that G is a VE graph. $\square$

Theorem 13. *Let G be a VV graph. Then G is a VE graph.*

Proof. Let $C = \{v_1, \dots, v_{2j+1}\}$ be an odd cycle of G. By hypothesis, as G is a VV graph, there is a vertex v_i that dominates a vertex v_k ($v_i \neq v_k$). We may suppose that $v_k \neq v_{i-1}$, so v_i dominates $f = (v_k, v_{k+1})$, which implies that G is a VE graph. □

Finally, we can determine that these classes of graphs are balanced graphs.

Theorem 14. *Let G be a VE graph. Then G is a balanced graph.*

Proof. Suppose that A_G is not a balanced matrix. So, we have the matrix of Figure 3 as a submatrix A' in A_G, where $M_1, \dots, M_{2k+1}$ are cliques of G and $v_1, \dots, v_{2k+1}$ are vertices of G. Then $v_1, \dots, v_{2k+1}$ is an odd cycle of G and M_i is a clique that contains the edge (v_i, v_{i+1}) ($M_i \in M(v_i, v_{i+1})$). But M_i does not contain another vertex v_j of the cycle, otherwise there would be a 1 in the position (i, j) of A'. So $M_i \notin M(v_j)$ for $j \neq i, i+1$. This fact implies that $N_G[(v_i, v_{i+1})] \not\subseteq N_G[v_j]$ for $j \neq i, i+1$, for any edge (v_i, v_{i+1}) of the cycle, thus G is not a VE graph. □

Corollary 5. *VE, EE, VV and EV graphs are perfect graphs.*

Note: Figure 5 shows examples of minimal graphs belonging to the possible intersections defined by the inclusions among these classes. The examples can be checked with no difficulty. We can see in this figure that the inclusions are proper.

Remark 2. Bipartite graphs are EV graphs.

Remark 3. VE, EE, VV and EV graphs are hereditary classes of graphs.

4.2. Matrix characterizations

Let $e_1, \dots, e_m$ and $v_1, \dots, v_n$ be the edges and vertices of a graph G, respectively. Denote by w_{1i} and w_{2i} the endpoints of the edge e_i. We define two matrices in $\{0,1\}^{m \times n}$:

- $A_{VE}(G)$, whose entry (i, j) is 1 if $N_G[e_i] \subseteq N_G[v_j]$, and 0 otherwise.
- $A_{VV}(G)$, whose entry (i, j) is 1 if $N_G[w_{1i}] \subseteq N_G[v_j]$ or $N_G[w_{2i}] \subseteq N_G[v_j]$, and 0 otherwise.

Clearly, both matrices can be constructed in polynomial time.

Theorem 15. *A graph G is a VE graph if and only if $A_{VE}(G)$ is a balanced matrix.*

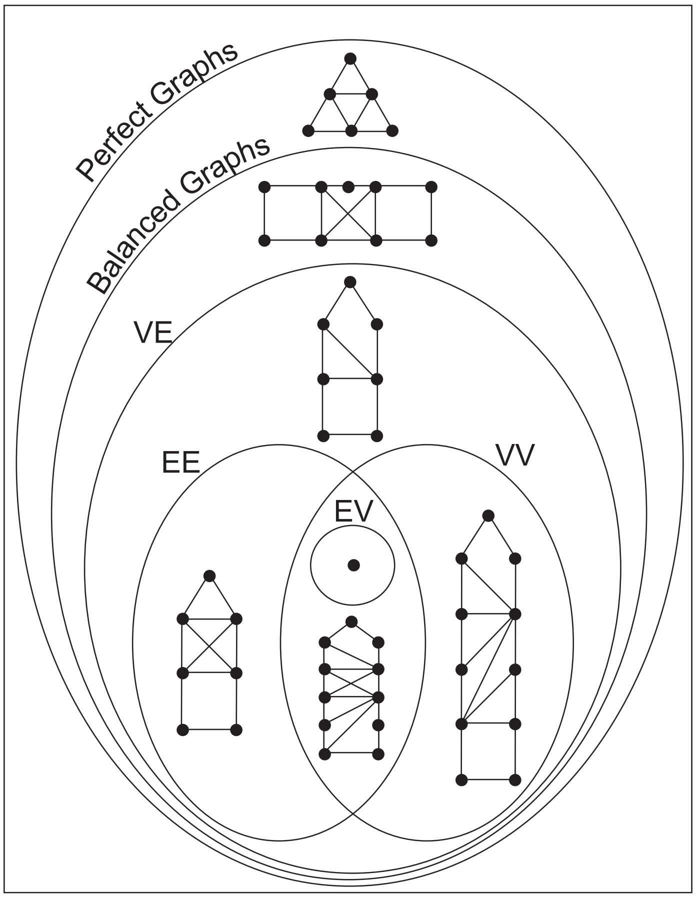

Fig. 5. Intersection between all the classes

	v_1	v_2	v_3	$\dots$	v_{2k+1}
e_1	1	1	0	$\dots$	0
e_2	0	1	1	$\dots$	0
e_3	0	0	1	$\dots$	0
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
e_{2k+1}	1	0	0	$\dots$	1

Fig. 6. Vertex-edge incidence matrix of an odd cycle

Proof. $\Rightarrow$) Suppose that $A_{VE}(G)$ is not a balanced matrix. So, we have the following submatrix A' in $A_{VE}(G)$, where $e_1, \ldots, e_{2k+1}$ are edges of G and $v_1, \ldots, v_{2k+1}$ are vertices of G:

Let $1 \leq i \leq 2k+1$. Since $N_G[e_i] \subseteq N_G[v_i] \cap N_G[v_{i+1}]$, v_i and v_{i+1} are adjacent, and then $v_1, \ldots, v_{2k+1}$ is an odd cycle of G. Let f_i be the edge (v_i, v_{i+1}). Then $N_G[e_i] \subseteq N_G[f_i]$. So, if the vertex v_j dominates the edge f_i, then it also dominates the edge e_i and, therefore, there must be a 1 in the position (i,j) of A'. So the vertex v_j does not dominate the edge f_i for $j \neq i, i+1$, for any edge f_i of the cycle. Thus G is not a VE graph.

$\Leftarrow$) Suppose that G is not a VE graph. Then there is an odd cycle $C = \{v_1, \ldots, v_{2k+1}\}$ such that, for any $e_i = (v_i, v_{i+1})$ and any $j \neq i, i+1$, $N_G[e_i] \not\subseteq N_G[v_j]$.

Now, if we choose the rows of $A_{VE}(G)$ corresponding to $e_1, \ldots, e_{2k+1}$ and the columns of $A_{VE}(G)$ corresponding to $v_1, \ldots, v_{2k+1}$, we have a vertex-edge incidence matrix of an odd cycle as a submatrix of $A_{VE}(G)$, so it is not a balanced matrix. $\square$

Corollary 6. *There is a polynomial-time recognition algorithm for VE graphs.*

Theorem 16. *A graph G is a VV graph if and only if $A_{VV}(G)$ is a balanced matrix.*

Proof. $\Rightarrow$) Suppose that $A_{VV}(G)$ is not a balanced matrix. So, we have the matrix of Figure 6 as a submatrix A' in $A_{VV}(G)$, where $e_1, \ldots, e_{2k+1}$ are edges of G and $v_1, \ldots, v_{2k+1}$ are vertices of G.

Let $1 \leq i \leq 2k+1$. By definition of $A_{VV}(G)$, $N_G[e_i] \subseteq N_G[v_i] \cap N_G[v_{i+1}]$, and therefore v_i and v_{i+1} are adjacent. Then $v_1, \ldots, v_{2k+1}$ is an odd cycle of G.

Note that, if the vertex v_j dominates the vertex v_i, there must be a 1 in the position (i,j) of A' and a 1 in the position $(i-1,j)$ of A' (the sums must be understood modulo 2k+1). However, the latter does not occur. So the vertex v_j does not dominate the vertex v_i for any $j \neq i$. Thus G is not a VV graph.

$\Leftarrow$) Suppose that G is not a VV graph. Then there is an odd cycle $C = \{v_1, \ldots, v_{2k+1}\}$ such that, for any $i \neq j$, $N_G[v_i] \not\subseteq N_G[v_j]$. If we choose the rows of $A_{VV}(G)$ corresponding to $e_1, \ldots, e_{2k+1}$ and the columns of $A_{VV}(G)$ corresponding to $v_1, \ldots, v_{2k+1}$, we have a vertex-edge incidence matrix of an odd cycle as a submatrix of $A_{VV}(G)$, so it is not a balanced matrix. $\square$

Corollary 7. *There is a polynomial-time recognition algorithm for VV graphs.*

4.3. A combinatorial algorithm for the maximum stable set in VV graphs

The maximum stable set problem can be solved in polynomial time for perfect graphs [20] (and in consequence for balanced graphs and its subclasses too), but the algorithm is based on linear programming. We present here a polynomial time purely combinatorial algorithm (i.e. non LP-based) for the problem of determining the maximum stable set in VV graphs.

Lemma 2. *Let G be a graph and v, w two vertices of G such that v dominates w. Then there exists a maximum stable set S of G such that v does not belong to S.*

Proof. Let S be a maximum stable set in G. If v does not belong to S, the lemma holds. Otherwise, w cannot belong to S because it is adjacent to v. As v dominates w, $S \setminus \{v\} \cup \{w\}$ is a maximum stable set that does not contain v. $\square$

Theorem 17. *There exists a polynomial time combinatorial algorithm to find a maximum stable set for VV graphs.*

Proof. Let G be a VV graph. If there exists a vertex v that dominates another vertex w, then remove v. This procedure is repeated until no more dominating vertices exist. We obtain an induced subgraph G' that can be constructed in polynomial time. As VV graphs are hereditary, G' lies in this class. So, G' has no odd cycle (and in consequence is a bipartite graph). By Lemma 2, a maximum stable set in G' is a maximum stable set in G. Such a set can be found in $O(n^{5/2})$ time [23]. $\square$

5. Clique graphs of balanced graphs

Clique graphs of several classes of graphs have been already characterized. Trees, interval graphs, chordal graphs, block graphs, clique-Helly graphs and Helly circular-arc graphs are some of them [30]. In this section we see that the class of balanced graphs and the class of totally unimodular graphs are fixed classes under the clique operator, i.e. $K(\mathit{BALANCED}) = \mathit{BALANCED}$ and $K(\mathit{TOTALLY\ UNIMODULAR}) = \mathit{TOTALLY\ UNIMODULAR}$, and finally we present a characterization of clique graphs of VE, EE, VV and EV graphs.

First some definitions and lemmas are needed. Let A_G^t be the transpose matrix of A_G. The following lemma is clear.

Lemma 3. *Let G be a clique-Helly graph. Then $A_{K(G)}$ is the submatrix of A_G^t obtained by removing the included rows.*

Define the graph $H(G)$ where $V(H(G)) = \{q_1, \dots, q_k, w_1, \dots, w_n\}$, each q_i corresponds to the clique M_i of G, and each w_i corresponds to the vertex v_i of G. The edges of $H(G)$ are the following: the vertices $q_1, \dots, q_k$ induce the graph $K(G)$, the vertices $w_1, \dots, w_n$ induce a stable set and w_j is adjacent to q_i if and only if v_j belongs to the clique M_i in G.

Theorem 18. [22] *Let G be a clique-Helly graph and $H(G)$ as defined above. Then the cliques of $H(G)$ are induced by $N_G[w_i]$ for each i, w_i is a simplicial vertex of $H(G)$ for every i, and $K(H(G)) = G$.*

Let $A \in R^{n\times m}$ and $B \in R^{n\times k}$ be two matrices. We define the matrix $A|B \in R^{n\times(m+k)}$ by $(A|B)(i,j) = A(i,j)$ for $i = 1, \dots, n$, $j = 1, \dots, m$ and $(A|B)(i, m+j) = B(i,j)$ for $i = 1, \dots, n$, $j = 1, \dots, k$. Let I_n be the $n \times n$ identity matrix.

Theorem 19. *Let G be a clique-Helly graph and $|V(G)| = n$. Then $A_{H(G)} = A^t_G|I_n$.*

Proof. It follows immediately from the previous theorem. □

From Lemma 3 we can deduce the following result, proved in [5] too:

Theorem 20. *If G is a balanced graph then $K(G)$ is also balanced.*

Theorem 21. *A graph G is balanced if and only if G is clique-Helly and $H(G)$ is balanced.*

Proof. ⇒) If G is a balanced graph, then by Corollary 1, G is a clique-Helly graph. So, we have that $A_{H(G)} = A^t_G|I_n$ (Theorem 19), and A_G is balanced, so A^t_G is balanced. On the other hand, all the columns of the vertex-edge incidence matrix of an odd cycle have two nonzero entries, so $A_{H(G)}$ is balanced.

⇐) If G is a clique-Helly graph and $H(G)$ is balanced, $G = K(H(G))$ (Theorem 18) and then G is balanced (Theorem 20). □

The following corollary, mentioned in [26], follows from Theorem 20, Corollary 1 and Theorem 21.

Corollary 8. *The class of balanced graphs is fixed under K, that is, $K(BALANCED) = BALANCED$.*

Next, we show that similar results hold for the class of totally unimodular graphs.

Theorem 22. *If G is a totally unimodular graph then $K(G)$ is also totally unimodular.*

Proof. If G is a totally unimodular graph then G is a balanced graph and then G is a clique-Helly graph (Corollary 1). So Lemma 3 holds. If A_G is a totally unimodular matrix, then A_G^t is totally unimodular too, since for every square matrix M, $det(M) = det(M^t)$. And every submatrix of a totally unimodular matrix is totally unimodular. So, $A_{K(G)}$ is a totally unimodular matrix. □

Theorem 23. *A graph G is totally unimodular if and only if G is clique-Helly and $H(G)$ is totally unimodular.*

Proof. $\Rightarrow$) If G is a totally unimodular graph then G is a balanced graph and consequently G is a clique-Helly graph (Corollary 1). We have that $A_{H(G)} = A_G^t|I_n$ (Theorem 19), and A_G is totally unimodular, so A_G^t is totally unimodular. Every square submatrix M of $A_{H(G)}$ can be written as $M = M_1|M_2$, where M_1 is a submatrix of A_G^t and M_2 is a submatrix of I_n. So, using determinant properties, M is singular or $det(M) = \pm det(M_3)$, where M_3 is a square submatrix of M_1. Then, in both cases, $det(M) = 0$ or ± 1. Therefore $H(G)$ is totally unimodular.

$\Leftarrow$) If G is a clique-Helly graph and $H(G)$ is totally unimodular, $G = K(H(G))$ (Theorem 18) and then G is totally unimodular (Theorem 22). □

Corollary 9. *The class of totally unimodular graphs is fixed under K, i.e., $K(TOTALLY\ UNIMODULAR) = TOTALLY\ UNIMODULAR$.*

Finally, we present a characterization of clique graphs of VE, EE, VV and EV graphs.

Let $S = \{M_1, \ldots, M_{2k+1}\}$ be an odd set of cliques of G, where M_r intersects M_{r+1} for $r = 1, \ldots, 2k+1$ (all the sums must be understood modulo 2k+1).

A graph G is a *dually EE* graph (DEE graph) if for any such a set S there exist i, j, $1 \le i, j \le 2k+1$, $i \ne j$, such that $M_i \cap M_{i+1} \subseteq M_j \cap M_{j+1}$.

A graph G is a *dually VE* graph (DVE graph) if for any such a set S there exist i, j, $1 \le i, j \le 2k+1$, $i \ne j$, $i+1 \ne j$, such that $M_i \cap M_{i+1} \subseteq M_j$.

Theorem 24. *Let G be a DEE graph. Then G is a DVE graph.*

Proof. Let $S = \{M_1, \ldots, M_{2k+1}\}$ a set of cliques of G, where M_i intersects M_{i+1} for $i = 1, \ldots, 2k+1$. By hypothesis, as G is a DEE graph, there are cliques $M_i, M_{i+1}, M_j, M_{j+1}$ such that $M_i \cap M_{i+1} \subseteq M_j \cap M_{j+1}$ ($i \ne j$). So $M_i \cap M_{i+1} \subseteq M_j$, and if $i+1 = j$ then $i \ne j+1, i+1 \ne j+1$ and $M_i \cap M_{i+1} \subseteq M_{j+1}$, which implies that G is a DVE graph. □

Theorem 25. *Let G be a DVE graph. Then G is a balanced graph.*

Proof. Suppose that A_G is not a balanced matrix. So, we have the matrix of Figure 3 as a submatrix A' in A_G, where $M_1, \ldots, M_{2k+1}$ are cliques of G and $v_1, \ldots, v_{2k+1}$ are vertices of G. Then $\{M_1, \ldots, M_{2k+1}\}$ is an odd set of cliques of G where M_i intersects M_{i+1} for $i = 1, \ldots, 2k+1$. On the other hand, v_i is a vertex that belongs to $M_i \cap M_{i+1}$ but v_i does not belong to another clique M_j of the set, otherwise there would be a 1 in the position (j, i) of A'. So $v_i \notin M_j$ for $j \neq i, i+1$. This fact implies that $M_i \cap M_{i+1} \not\subseteq M_j$ for $j \neq i, i+1$, for any $i = 1, \ldots, 2k+1$, thus G is not a DVE graph. □

Theorem 26. *Let G be a graph.*

- *If G is a DVE graph then $K(G)$ is VE.*
- *If G is a DEE graph then $K(G)$ is EE.*
- *If G is a VE graph then $K(G)$ is DVE.*
- *If G is a EE graph then $K(G)$ is DEE.*

Proof. Let G be a graph. Classes DVE, DEE, VE and EE are subclasses of balanced graphs, and balanced graphs are clique-Helly. So, if G belongs to some of these classes, then G is a clique-Helly graph. The vertices of $K(G)$ are the cliques of G, and by Lemma 3 we know that the cliques of $K(G)$ are some $M(v)$ with $v \in V(G)$.

Let $\{M_1, \ldots, M_{2k+1}\}$ be an odd cycle in $K(G)$, then M_i intersects M_{i+1} in G, for $i = 1, \ldots, 2k+1$.

If G is a DVE graph, there are cliques M_i, M_{i+1}, M_j such that $M_i \cap M_{i+1} \subseteq M_j$ $(i, i+1 \neq j)$. Let $M(v)$ be a clique of $K(G)$ that contains M_i and M_{i+1}. Then, in G, v lies in $M_i \cap M_{i+1}$ implying that v is in M_j and therefore $M(v)$ contains M_j too. So, in $K(G)$, the vertex M_j dominates the edge (M_i, M_{i+1}) and, as a consequence, $K(G)$ is in VE.

If G is a DEE graph, there are cliques $M_i, M_{i+1}, M_j, M_{j+1}$ such that $M_i \cap M_{i+1} \subseteq M_j \cap M_{j+1}$ $(i \neq j)$. Let $M(v)$ be a clique of $K(G)$ that contains M_i and M_{i+1}, then, in G, v lies in $M_i \cap M_{i+1}$ implying that v is in $M_j \cap M_{j+1}$ and therefore $M(v)$ contains M_j and M_{j+1} too. So, in $K(G)$, the edge (M_j, M_{j+1}) dominates the edge (M_i, M_{i+1}) and, in consequence, $K(G)$ is in EE.

Now, let $\{M(v_1), \ldots, M(v_{2k+1})\}$ be an odd set of cliques in $K(G)$, where $M(v_i)$ intersects $M(v_{i+1})$ for $i = 1, \ldots, 2k+1$. Then for each i there exists a clique M_i of G such that v_i and v_{i+1} belong to M_i, and then v_i and v_{i+1} are adjacent in G, so $v_1, \ldots, v_{2k+1}$ is an odd cycle in G.

If G is in VE, there is a vertex v_j of the cycle that dominates the edge (v_i, v_{i+1}) with $j \neq i, i+1$. Let M be a vertex of $K(G)$, M lies in $M(v_i) \cap M(v_{i+1})$ in $K(G)$, v_i and v_{i+1} belong to M in G, and therefore v_j belongs to M too. So $M \in M(v_j)$, and in consequence $M(v_i) \cap M(v_{i+1}) \subseteq M(v_j)$. Then $K(G)$ is a DVE graph.

If G is in EE, there is an edge (v_j, v_{j+1}) of the cycle that dominates the edge (v_i, v_{i+1}) with $j \neq i$. Let M be a vertex of $K(G)$, M lies in $M(v_i) \cap M(v_{i+1})$ in $K(G)$, v_i and v_{i+1} belong to M in G, and therefore v_j and v_{j+1} belong to M too. So $M \in M(v_j) \cap M(v_{j+1})$, and in consequence $M(v_i) \cap M(v_{i+1}) \subseteq M(v_j) \cap M(v_{j+1})$. Then $K(G)$ is a DEE graph. $\square$

Theorem 27. *Let G be a clique-Helly graph.*

- *G is a DVE graph if and only if $H(G)$ is VE.*
- *G is a DEE graph if and only if $H(G)$ is EE.*
- *G is a VE graph if and only if $H(G)$ is DVE.*
- *G is a EE graph if and only if $H(G)$ is DEE.*

Proof. Let G be a clique-Helly graph and H(G) as defined in Theorem 18, with $V(H(G)) = \{q_1, \ldots, q_k, w_1, \ldots, w_n\}$, each q_i corresponds to the clique M_i of G, and each w_i corresponds to the vertex v_i of G. By Theorem 18, the cliques of $H(G)$ are $N_{H(G)}[w_i]$ for each i. Then w_i and all its incident edges are dominated between themselves, and every vertex in $N_{H(G)}(w_i)$ dominates w_i and all its incident edges.

Let C be an odd cycle in $H(G)$. If there is a vertex w_i in C, then C contains an edge that dominates another edge, and a vertex that dominates an edge non incident to it.

If there is not such a vertex, C is an odd cycle $\{q_{r_1}, \ldots, q_{r_{2s+1}}\}$ that corresponds to an odd set of cliques $\{M_{r_1}, \ldots, M_{r_{2s+1}}\}$ of G, such that M_{r_i} intersects $M_{r_{i+1}}$ for $i = 1, \ldots, 2s+1$.

If G is a DVE graph, there are cliques $M_{r_i}, M_{r_{i+1}}, M_{r_j}$ such that $M_{r_i} \cap M_{r_{i+1}} \subseteq M_{r_j}$ $(i, i+1 \neq j)$. Let $N_{H(G)}[w_l]$ be a clique of $H(G)$ that contains q_{r_i} and $q_{r_{i+1}}$. Then, in G, v_l lies in $M_{r_i} \cap M_{r_{i+1}}$ implying that v_l is in M_{r_j} and therefore, in $H(G)$, $N_{H(G)}[w_l]$ contains q_{r_j} too. So, in $H(G)$, the vertex q_{r_j} dominates the edge $(q_{r_i}, q_{r_{i+1}})$ and, in consequence, $H(G)$ is VE.

If G is a DEE graph, there are cliques $M_{r_i}, M_{r_{i+1}}, M_{r_j}, M_{r_{j+1}}$ such that $M_{r_i} \cap M_{r_{i+1}} \subseteq M_{r_j} \cap M_{r_{j+1}}$ $(i \neq j)$. Let $N_{H(G)}[w_l]$ be a clique of $H(G)$ that contains M_{r_i} and $M_{r_{i+1}}$. Then, in G, v_l belongs to $M_{r_i} \cap M_{r_{i+1}}$ implying that v_l belongs to $M_{r_j} \cap M_{r_{j+1}}$. Therefore, in $H(G)$, $N_{H(G)}[w_l]$ contains q_{r_j} and $q_{r_{j+1}}$ too. So, in $H(G)$, the edge $(q_{r_j}, q_{r_{j+1}})$ dominates the edge $(q_{r_i}, q_{r_{i+1}})$ and, in consequence, $H(G)$ is EE.

Now, let $\{N_{H(G)}[w_{r_1}], \ldots, N_{H(G)}[w_{r_{2s+1}}]\}$ be an odd set of cliques in $H(G)$, where $N_{H(G)}[w_{r_i}]$ intersects $N_{H(G)}[w_{r_{i+1}}]$ for $i = 1, \ldots, 2s+1$. Then for each i there exists a vertex $q \in N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}]$. So v_{r_i} and $v_{r_{i+1}}$ belong to the corresponding clique M of G, and then v_{r_i} and $v_{r_{i+1}}$ are adjacent in G, so $v_{r_1}, \ldots, v_{r_{2s+1}}$ is an odd cycle in G.

If G is a VE graph, there is a vertex v_{r_j} of the cycle that dominates the edge $(v_{r_i}, v_{r_{i+1}})$ with $j \neq i, i+1$. Let q_l be a vertex of $H(G)$, q_l lies in $N_{H(G)}[w_{r_i}] \cap$

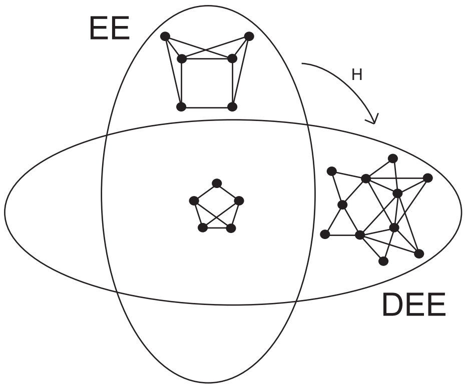

Fig. 7. Intersection between the dual classes EE and DEE

$N_{H(G)}[w_{r_{i+1}}]$ in $H(G)$, v_i and v_{i+1} belong to M_l in G, and therefore v_j belongs to M_l too. So q_l belongs to $N_{H(G)}[w_{r_j}]$, and in consequence $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}] \subseteq N_{H(G)}[w_{r_j}]$. Then $H(G)$ is DVE.

If G is a EE graph, there is an edge $(v_{r_j}, v_{r_{j+1}})$ of the cycle that dominates the edge $(v_{r_i}, v_{r_{i+1}})$ with $j \neq i$. Let q_l be a vertex of $H(G)$, q_l lies in $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}]$ in $H(G)$, v_{r_i} and $v_{r_{i+1}}$ belong to M_l in G, and therefore v_{r_j} and $v_{r_{j+1}}$ belong to M_l too. So q_l belongs to $N_{H(G)}[w_{r_j}] \cap N_{H(G)}[w_{r_{j+1}}]$ in $H(G)$, and in consequence $N_{H(G)}[w_{r_i}] \cap N_{H(G)}[w_{r_{i+1}}] \subseteq N_{H(G)}[w_{r_j}] \cap N_{H(G)}[w_{r_{j+1}}]$. Then $H(G)$ is DEE.

The converse properties follow from Theorem 18 and Theorem 26 applied to $H(G)$. □

Corollary 10. *$K(DEE) = EE$ and $K(EE) = DEE$.*

Corollary 11. *$K(DVE) = VE$ and $K(VE) = DVE$.*

The following result by Escalante is needed to analyse clique graphs of VV graphs.

Theorem 28. [16] *If G is a clique-Helly graph then $K^2(G)$ is the subgraph of G obtained by removing the dominated vertices.*

Theorem 29. *Let G be a graph. If G is a VV graph then $K^2(G)$ is a bipartite graph.*

Proof. If G is a VV graph then G is clique-Helly (Corollary 1). Every odd cycle of G has a dominated vertex, and therefore, by Theorem 28, $K^2(G)$ is a bipartite graph. □

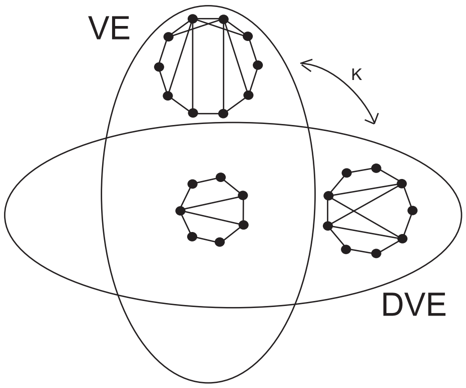

Fig. 8. Intersection between the dual classes VE and DVE

Theorem 30. *Let G be a graph. Then $K(G)$ is a bipartite graph if and only if G is a clique-Helly graph and $H(G)$ is an EV graph.*

Proof. $\Rightarrow$) Let G be a graph, $V(G) = \{v_1, \ldots, v_n\}$ and $M(G) = \{M_1, \ldots, M_k\}$. Since $K(G)$ is a bipartite graph, G is clique-Helly because any set of pairwise intersecting cliques has at most two elements. Clearly, $V(H(G)) = V(K(G)) \cup \{w_1, \ldots, w_n\}$ as in the definition of $H(G)$. Also, $K(G)$ is a bipartite graph and by the definition of $H(G)$, every odd cycle C of $H(G)$ must contain a vertex w_i from $\{w_1, \ldots, w_n\}$. By Theorem 18, w_i is a simplicial vertex, so the edges of C incident to w_i dominate the vertex w_i, and then $H(G)$ is an EV graph.

$\Leftarrow$) If G is a clique-Helly graph and $H(G)$ is an EV graph, it is a VV graph too. So by Theorem 29, $K^2(H(G)) = K(G)$ is a bipartite graph. □

Corollary 12. $K^2(VV) = K^2(EV) =$ *the class of bipartite graphs.*

Proof. We will prove that $K^2(EV) \subseteq K^2(VV) \subseteq \mathit{BIPARTITE} \subseteq K^2(EV)$ and therefore the three classes are the same. The first inclusion holds because $EV \subseteq VV$. The second inclusion follows from Theorem 29. Now, for every bipartite graph G we have that $K(H(G)) = G$ and by Theorem 30 applied to $H(G)$, $H^2(G)$ is an EV graph and $K^2(H^2(G)) = G$. So the third inclusion holds too. □

The class $K^{-1}(\mathit{BIPARTITE})$ has been analysed and characterized by forbidden subgraphs in [28].

Corollary 13. $K(VV) = K(EV) = K^{-1}(\mathit{BIPARTITE})$.

Proof. Let G be a VV graph. By the last corollary, $K^2(G) = K(K(G))$ is bipartite so $K(G)$ belongs to $K^{-1}(\mathit{BIPARTITE})$. Therefore $K(EV) \subseteq K(VV) \subseteq$

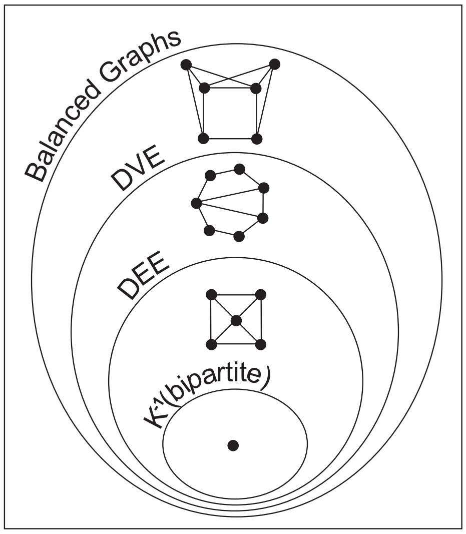

Fig. 9. Inclusion between the classes

$K^{-1}(BIPARTITE)$. On the other hand, let G be a graph belonging to $K^{-1}(BIPARTITE)$, then by Theorem 30 $H(G)$ is EV and $G = K(H(G))$. So $K^{-1}(BIPARTITE) \subseteq K(EV) \subseteq K(VV) \subseteq K^{-1}(BIPARTITE)$ and we have that the three sets are equal. □

As a consequence of this result, we deduce the existence of pure combinatorial algorithms to find a maximum clique-independent set and a minimum clique-transversal for VV graphs.

Corollary 14. *There exists a polynomial time combinatorial algorithm to find a maximum clique-independent set and a minimum clique-transversal for VV graphs.*

Proof. Let G be a VV graph. Then $K(G)$ belongs to $K^{-1}(BIPARTITE)$ and can be constructed in polynomial time. Moreover, a maximum clique-independent set of G can be obtained from a maximum stable set of $K(G)$, and a minimum clique-transversal of G can be constructed from a minimum clique covering of $K(G)$. Since the graphs $K^{-1}(BIPARTITE)$ are $K_{1,3}$-free [28] there exists a polynomial time combinatorial algorithm for maximum stable set in these graphs [29]. As $K(G)$ is also perfect, we can use the polynomial time combinatorial algorithm for minimum clique covering in $K_{1,3}$-free perfect graphs [24]. So, the result holds. □

Finally, we can see that $K^{-1}(BIPARTITE)$ graphs are a subclass of DEE.

Theorem 31. $K^{-1}(BIPARTITE) \subseteq DEE$.

Proof. Let $G \in K^{-1}(BIPARTITE)$. Suppose that there exists an odd set $S = \{M_1, \ldots, M_{2k+1}\}$ of cliques of G, where M_i intersects M_{i+1} for $i = 1, \ldots, 2k$ and M_{2k+1} intersects M_1. Then the corresponding vertices in $K(G)$ form an odd cycle, but $K(G)$ is a bipartite graph, so such a set does not exist, and G is DEE. $\square$

Note 1. In Figure 9, we can see that all these inclusions are proper.

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